Vectors and Planes

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Let $\mathbf{a} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$.

(a) Find $\mathbf{a}+\mathbf{b}$. **[1]**

(b) Find $|\mathbf{a}|$. **[2]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Find the angle between the vectors $\mathbf{u} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$. **[4]**

3. [Maximum mark: 5] *Difficulty:* 2/5

The line L has Cartesian equation $\dfrac{x+2}{4} = \dfrac{y-1}{3} = z-5$.

(a) Show that the point $(-2, 1, 5)$ lies on L. **[2]**

(b) Find a vector equation of L. **[3]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Find the angle between the vectors $\mathbf{d_1} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. **[5]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Consider the points $A(0,3,-2)$, $B(4,k,2)$ and $C(12,19,10)$, where $k \in \mathbb{R}$.

(a) Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, in terms of k where necessary. **[2]**

(b) Given that A, B and C lie on a straight line, find the value of k. **[4]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Show that L_1 and L_2 are not parallel. **[2]**

(b) Show that L_1 and L_2 do not intersect, and hence that they are skew. **[5]**

7. [Maximum mark: 8] *Difficulty:* 4/5

Consider the points $A(3,0,1)$, $B(1,2,0)$ and $C(2,1,3)$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. **[2]**

(b) Find a normal vector to the plane containing A, B and C. **[3]**

(c) Hence find the Cartesian equation of this plane. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Consider the plane $\Pi : x - 3y + 2z = 8$ and the point $P(0,2,3)$.

(a) Write down a normal vector to Π. **[1]**

(b) Find the distance from P to Π. **[4]**

(c) Find the coordinates of the foot of the perpendicular from P to Π. **[3]**

9. [Maximum mark: 9] *Difficulty:* 4/5

Consider the line $L : \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and the plane $\Pi : 3x + y - 2z = 8$.

(a) Find the coordinates of the point of intersection of L and Π. **[4]**

(b) Find the angle between L and Π. **[5]**

10. [Maximum mark: 13] *Difficulty:* 5/5

Challenge question. Consider the lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} k \\ 1 \\ -1 \end{pmatrix},$$

where $k \in \mathbb{R}$.

(a) Find the possible values of k when the acute angle between L_1 and L_2 is 60°. Give your answers in the form $\dfrac{p \pm q\sqrt{r}}{s}$, where $p, q, r, s \in \mathbb{Z}^+$. **[7]**

(b) Using the smaller of the two values of k found in part (a), determine whether L_1 and L_2 are parallel, skew, or intersecting. Justify your answer fully. **[6]**

Answer Key

1. (a) $\mathbf{a}+\mathbf{b}=\begin{pmatrix}6\\1\\-2\end{pmatrix}$. (b) $|\mathbf{a}|=\sqrt{16+4+1}=\sqrt{21}$.

2. $\mathbf{u}\cdot\mathbf{v}=2+6+2=10$; $|\mathbf{u}|=3$, $|\mathbf{v}|=\sqrt{14}$. $\cos\theta=\dfrac{10}{3\sqrt{14}}\Rightarrow\theta\approx 27.0^\circ$.

3. (a) At $(-2,1,5)$: $\dfrac{-2+2}{4}=0$, $\dfrac{1-1}{3}=0$, $5-5=0$; all equal, so the point lies on L. (b) Direction vector $(4,3,1)$; $\mathbf{r}=\begin{pmatrix}-2\\1\\5\end{pmatrix}+t\begin{pmatrix}4\\3\\1\end{pmatrix}$.

4. $\mathbf{d_1}\cdot\mathbf{d_2}=2+2+4=8$; $|\mathbf{d_1}|=3$, $|\mathbf{d_2}|=3$. $\cos\theta=\dfrac{8}{9}\Rightarrow\theta\approx 27.3^\circ$.

5. (a) $\overrightarrow{AB}=(4,k-3,4)$, $\overrightarrow{AC}=(12,16,12)$. (b) Since $\overrightarrow{AC}=3\overrightarrow{AB}$ (comparing first and third components: $12=3(4)$, $12=3(4)$), the second component requires $16=3(k-3)\Rightarrow k-3=\frac{16}{3}\Rightarrow k=\frac{25}{3}$.

6. (a) Direction vectors $(1,3,-2)$ and $(-1,1,2)$ are not scalar multiples of each other, so L_1, L_2 are not parallel. (b) Setting $L_1=L_2$: $2+s=-t$, $3s=2+t$, $1-2s=2t$. From the first: $t=-2-s$. Substituting into the second: $3s=2+(-2-s)\Rightarrow 3s=-s\Rightarrow 4s=0\Rightarrow s=0,t=-2$. Checking the third equation: $1-2(0)=1$ but $2(-2)=-4$; since $1\neq-4$, there is no consistent solution. As L_1, L_2 are neither parallel nor intersecting, they are skew. ■

7. (a) $\overrightarrow{AB}=(-2,2,-1)$, $\overrightarrow{AC}=(-1,1,2)$. (b) $\overrightarrow{AB}\times\overrightarrow{AC}=(5,5,0)$. (c) $5(x-3)+5y+0(z-1)=0\Rightarrow 5x+5y=15\Rightarrow x+y=3$.

8. (a) $\mathbf{n}=(1,-3,2)$. (b) Distance $=\dfrac{|0-6+6-8|}{\sqrt{1+9+4}}=\dfrac{8}{\sqrt{14}}\approx 2.14$. (c) $t=\dfrac{8-(0-6+6)}{14}=\dfrac{8}{14}=\dfrac{4}{7}$. Foot $=\left(0+\frac{4}{7},2-\frac{12}{7},3+\frac{8}{7}\right)=\left(\frac{4}{7},\frac{2}{7},\frac{29}{7}\right)$.

9. (a) Substituting into Π: $3(2t)+(3-t)-2(1+t)=8\Rightarrow 6t+3-t-2-2t=8\Rightarrow 3t+1=8\Rightarrow t=\frac{7}{3}$. Point $=\left(\frac{14}{3},\frac{2}{3},\frac{10}{3}\right)$. (b) $\mathbf{d}\cdot\mathbf{n}=6-1-2=3$; $|\mathbf{d}|=\sqrt{6}$, $|\mathbf{n}|=\sqrt{14}$. $\sin\theta=\dfrac{3}{\sqrt{84}}\Rightarrow\theta\approx 19.1^\circ$.

10. (a) $\mathbf{d_1}\cdot\mathbf{d_2}=2k+1-2=2k-1$; $|\mathbf{d_1}|=3$, $|\mathbf{d_2}|=\sqrt{k^2+2}$. Setting $\cos 60^\circ=\frac{1}{2}$: $\dfrac{|2k-1|}{3\sqrt{k^2+2}}=\dfrac{1}{2}\Rightarrow 2|2k-1|=3\sqrt{k^2+2}$. Squaring: $4(2k-1)^2=9(k^2+2)\Rightarrow 16k^2-16k+4=9k^2+18\Rightarrow 7k^2-16k-14=0$. By the quadratic formula: $k=\dfrac{16\pm\sqrt{256+392}}{14}=\dfrac{16\pm\sqrt{648}}{14}=\dfrac{16\pm18\sqrt{2}}{14}=\dfrac{8\pm9\sqrt{2}}{7}$. (b) Taking the smaller value $k=\dfrac{8-9\sqrt{2}}{7}\approx-0.675$: the direction vectors $(2,1,2)$ and $(k,1,-1)$ are not scalar multiples of each other (their component ratios are unequal), so L_1, L_2 are not parallel. Setting $L_1=L_2$: $1+2s=kt$, $s=1+t$, $2+2s=3-t$. From the second and third equations: $s=1+t$ and $2+2(1+t)=3-t\Rightarrow 4+2t=3-t\Rightarrow 3t=-1\Rightarrow t=-\frac{1}{3},s=\frac{2}{3}$. Checking the first equation with $k\approx-0.675$: LHS $=1+2(\frac{2}{3})=\frac{7}{3}\approx 2.33$; RHS $=kt\approx(-0.675)(-\frac{1}{3})\approx 0.225$. Since these are not equal, there is no consistent solution. As L_1, L_2 are neither parallel nor intersecting, they are skew.