Vectors and Planes

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Let $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$.

(a) Find $\mathbf{a}+\mathbf{b}$. **[1]**

(b) Find $|\mathbf{a}|$. **[2]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Find the angle between the vectors $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. **[4]**

3. [Maximum mark: 5] *Difficulty:* 2/5

The line L has Cartesian equation $\dfrac{x-1}{3} = \dfrac{y-4}{-2} = z+2$.

(a) Show that the point $(1,4,-2)$ lies on L. **[2]**

(b) Find a vector equation of L. **[3]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Find the angle between the vectors $\mathbf{d_1} = \begin{pmatrix} 5 \\ 0 \\ 12 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} 0 \\ 12 \\ 5 \end{pmatrix}$. **[5]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Consider the points $A(2,-1,0)$, $B(5,k,3)$ and $C(11,17,9)$, where $k \in \mathbb{R}$.

(a) Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, in terms of k where necessary. **[2]**

(b) Given that A, B and C lie on a straight line, find the value of k. **[4]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

(a) Show that L_1 and L_2 are not parallel. **[2]**

(b) Show that L_1 and L_2 do not intersect, and hence that they are skew. **[5]**

7. [Maximum mark: 8] *Difficulty:* 4/5

Consider the points $A(2,1,0)$, $B(1,3,-1)$ and $C(0,2,2)$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. **[2]**

(b) Find a normal vector to the plane containing A, B and C. **[3]**

(c) Hence find the Cartesian equation of this plane. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Consider the plane $\Pi : 3x + y - 4z = 10$ and the point $P(2,0,2)$.

(a) Write down a normal vector to Π. **[1]**

(b) Find the distance from P to Π. **[4]**

(c) Find the coordinates of the foot of the perpendicular from P to Π. **[3]**

9. [Maximum mark: 9] *Difficulty:* 4/5

Consider the line $L : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and the plane $\Pi : 2x - y + 3z = 11$.

(a) Find the coordinates of the point of intersection of L and Π. **[4]**

(b) Find the angle between L and Π. **[5]**

10. [Maximum mark: 15] *Difficulty:* 5/5

Challenge question. Consider the points $D(1,2,0)$, $E(3,0,1)$ and $F(2,3,2)$.

(a) Find $\overrightarrow{DE}$ and $\overrightarrow{DF}$. **[2]**

(b) Find a normal vector to the plane Π containing D, E and F. **[3]**

(c) Find the Cartesian equation of Π. **[3]**

(d) Find the distance from the origin $(0,0,0)$ to Π. **[3]**

(e) Find the coordinates of the point on Π that is closest to the origin. **[4]**

Answer Key

1. (a) $\mathbf{a}+\mathbf{b}=\begin{pmatrix}2\\3\\2\end{pmatrix}$. (b) $|\mathbf{a}|=\sqrt{9+1+4}=\sqrt{14}$.

2. $\mathbf{u}\cdot\mathbf{v}=2-2-4=-4$; $|\mathbf{u}|=3$, $|\mathbf{v}|=3$. $\cos\theta=-\dfrac{4}{9}\Rightarrow\theta\approx116.4°$.

3. (a) At $(1,4,-2)$: $\dfrac{1-1}{3}=0$, $\dfrac{4-4}{-2}=0$, $-2+2=0$; all equal, so the point lies on L. (b) Direction vector $(3,-2,1)$; $\mathbf{r}=\begin{pmatrix}1\\4\\-2\end{pmatrix}+t\begin{pmatrix}3\\-2\\1\end{pmatrix}$.

4. $\mathbf{d_1}\cdot\mathbf{d_2}=0+0+60=60$; $|\mathbf{d_1}|=13$, $|\mathbf{d_2}|=13$. $\cos\theta=\dfrac{60}{169}\Rightarrow\theta\approx69.2°$.

5. (a) $\overrightarrow{AB}=(3,k+1,3)$, $\overrightarrow{AC}=(9,18,9)$. (b) Since $\overrightarrow{AC}=3\overrightarrow{AB}$ (comparing first and third components: $9=3(3)$, $9=3(3)$), the second component requires $18=3(k+1)\Rightarrow k+1=6\Rightarrow k=5$.

6. (a) Direction vectors $(2,-1,1)$ and $(1,2,-1)$ are not scalar multiples of each other, so L_1, L_2 are not parallel. (b) Setting $L_1=L_2$: $2s=1+t$, $1-s=2t$, $2+s=3-t$. From the first and third: $s=\frac{1+t}{2}$ and $s=1-t$, so $\frac{1+t}{2}=1-t\Rightarrow1+t=2-2t\Rightarrow3t=1\Rightarrow t=\frac{1}{3},s=\frac{2}{3}$. Checking the second equation: $1-\frac{2}{3}=\frac{1}{3}$ but $2(\frac{1}{3})=\frac{2}{3}$; since $\frac{1}{3}\neq\frac{2}{3}$, there is no consistent solution. As L_1, L_2 are neither parallel nor intersecting, they are skew. ■

7. (a) $\overrightarrow{AB}=(-1,2,-1)$, $\overrightarrow{AC}=(-2,1,2)$. (b) $\overrightarrow{AB}\times\overrightarrow{AC}=(5,4,3)$. (c) $5(x-2)+4(y-1)+3z=0\Rightarrow 5x+4y+3z=14$.

8. (a) $\mathbf{n}=(3,1,-4)$. (b) Distance $=\dfrac{|3(2)+0-4(2)-10|}{\sqrt{9+1+16}}=\dfrac{12}{\sqrt{26}}\approx2.35$. (c) $t=\dfrac{10-(6+0-8)}{26}=\dfrac{12}{26}=\dfrac{6}{13}$. Foot $=\left(2+\frac{18}{13},0+\frac{6}{13},2-\frac{24}{13}\right)=\left(\frac{44}{13},\frac{6}{13},\frac{2}{13}\right)$.

9. (a) Substituting into Π: $2(1+2t)-(-1+t)+3(2+t)=11\Rightarrow2+4t+1-t+6+3t=11\Rightarrow9+6t=11\Rightarrow t=\frac{1}{3}$. Point $=\left(\frac{5}{3},-\frac{2}{3},\frac{7}{3}\right)$. (b) $\mathbf{d}\cdot\mathbf{n}=4-1+3=6$; $|\mathbf{d}|=\sqrt{6}$, $|\mathbf{n}|=\sqrt{14}$. $\sin\theta=\dfrac{6}{\sqrt{84}}\Rightarrow\theta\approx40.9°$.

10. (a) $\overrightarrow{DE}=(2,-2,1)$, $\overrightarrow{DF}=(1,1,2)$. (b) $\overrightarrow{DE}\times\overrightarrow{DF}=(-5,-3,4)$. (c) $-5(x-1)-3(y-2)+4z=0\Rightarrow-5x-3y+4z=-11\Rightarrow5x+3y-4z=11$. (d) Distance $=\dfrac{|5(0)+3(0)-4(0)-11|}{\sqrt{25+9+16}}=\dfrac{11}{\sqrt{50}}=\dfrac{11}{5\sqrt{2}}\approx1.56$. (e) $t=\dfrac{11-0}{50}=\dfrac{11}{50}$ (using normal $(5,3,-4)$ from the origin). Point $=\left(\frac{55}{50},\frac{33}{50},-\frac{44}{50}\right)=\left(\frac{11}{10},\frac{33}{50},-\frac{22}{25}\right)$.