Vectors and Planes

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

---

1. [Maximum mark: 3] *Difficulty:* 1/5

Let $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$.

(a) Find $\mathbf{a}+\mathbf{b}$. **[1]**

(b) Find $|\mathbf{a}|$. **[2]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Find the angle between the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. **[4]**

3. [Maximum mark: 5] *Difficulty:* 2/5

The line L has Cartesian equation $\dfrac{x-3}{2} = \dfrac{y+2}{-1} = z-1$.

(a) Show that the point $(3,-2,1)$ lies on L. **[2]**

(b) Find a vector equation of L. **[3]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Find the angle between the vectors $\mathbf{d_1} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$. **[5]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Consider the points $A(1,2,3)$, $B(4,8,k)$ and $C(7,14,15)$, where $k \in \mathbb{R}$.

(a) Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, in terms of k where necessary. **[2]**

(b) Given that A, B and C lie on a straight line, find the value of k. **[4]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

(a) Show that L_1 and L_2 are not parallel. **[2]**

(b) Show that L_1 and L_2 do not intersect, and hence that they are skew. **[5]**

7. [Maximum mark: 8] *Difficulty:* 4/5

Consider the points $A(1,0,2)$, $B(3,-1,1)$ and $C(2,2,0)$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. **[2]**

(b) Find a normal vector to the plane containing A, B and C. **[3]**

(c) Hence find the Cartesian equation of this plane. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Consider the plane $\Pi : 2x-2y+z=9$ and the point $P(1,1,1)$.

(a) Write down a normal vector to Π. **[1]**

(b) Find the distance from P to Π. **[4]**

(c) Find the coordinates of the foot of the perpendicular from P to Π. **[3]**

9. [Maximum mark: 9] *Difficulty:* 4/5

Consider the line $L : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and the plane $\Pi : x+2y-2z=7$.

(a) Find the coordinates of the point of intersection of L and Π. **[4]**

(b) Find the angle between L and Π. **[5]**

10. [Maximum mark: 15] *Difficulty:* 5/5

Challenge question. Consider the points $A(2,1,-1)$, $B(k,3,0)$ and $C(8,5,1)$, where $k \in \mathbb{R}$.

(a) Write down $\overrightarrow{AB}$ and $\overrightarrow{AC}$, in terms of k where necessary. **[2]**

(b) Given that A, B and C lie on a straight line, show that $k=5$. **[3]**

For $k=5$, let L_1 be the line through A, B and C.

(c) Find a vector equation of L_1. **[3]**

Consider a second line L_2 with equation $\dfrac{x-5}{1} = \dfrac{y-2}{-1} = z-3$.

(d) Show that L_1 and L_2 are skew. **[7]**

Answer Key

1. (a) $\mathbf{a}+\mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$. (b) $|\mathbf{a}| = \sqrt{4+1+9} = \sqrt{14}$.

2. $\mathbf{u}\cdot\mathbf{v} = 2-2+4 = 4$; $|\mathbf{u}| = 3$, $|\mathbf{v}| = 3$. $\cos\theta = \dfrac{4}{9} \Rightarrow \theta \approx 63.6°$.

3. (a) At $(3,-2,1)$: $\dfrac{3-3}{2} = 0$, $\dfrac{-2+2}{-1} = 0$, $1-1=0$; all equal, so the point lies on L. (b) Direction vector $(2,-1,1)$; $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

4. $\mathbf{d_1}\cdot\mathbf{d_2} = 0+0+12 = 12$; $|\mathbf{d_1}| = 5$, $|\mathbf{d_2}| = 5$. $\cos\theta = \dfrac{12}{25} = 0.48 \Rightarrow \theta \approx 61.3°$.

5. (a) $\overrightarrow{AB} = (3,6,k-3)$, $\overrightarrow{AC} = (6,12,12)$. (b) Since $\overrightarrow{AC} = 2\overrightarrow{AB}$ (comparing first two components: $6 = 2(3)$, $12 = 2(6)$), the third component requires $12 = 2(k-3) \Rightarrow k-3 = 6 \Rightarrow k = 9$.

6. (a) Direction vectors $(1,-1,2)$ and $(2,1,-1)$ are not scalar multiples of each other, so L_1, L_2 are not parallel. (b) Setting $L_1 = L_2$: $1+s = 2t$, $2-s = 3+t$, $2s = 1-t$. From the first two equations: $s = 2t-1$ and substituting, $2-(2t-1) = 3+t \Rightarrow 3-2t = 3+t \Rightarrow t = 0, s = -1$. Checking the third equation: $2(-1) = -2$ but $1-0 = 1$; since $-2 \neq 1$, there is no consistent solution. As L_1, L_2 are neither parallel nor intersecting, they are skew. ■

7. (a) $\overrightarrow{AB} = (2,-1,-1)$, $\overrightarrow{AC} = (1,2,-2)$. (b) $\overrightarrow{AB}\times\overrightarrow{AC} = (4,3,5)$. (c) $4(x-1)+3y+5(z-2) = 0 \Rightarrow 4x+3y+5z = 14$.

8. (a) $\mathbf{n} = (2,-2,1)$. (b) Distance $= \dfrac{|2(1)-2(1)+1-9|}{\sqrt{4+4+1}} = \dfrac{8}{3}$. (c) Foot $= P + t\mathbf{n}$ where $t = \dfrac{9-(2-2+1)}{9} = \dfrac{8}{9}$: foot $= \left(1+\frac{16}{9}, 1-\frac{16}{9}, 1+\frac{8}{9}\right) = \left(\frac{25}{9}, -\frac{7}{9}, \frac{17}{9}\right)$.

9. (a) Substituting into Π: $(2+t)+2(1-2t)-2(-1+2t) = 7 \Rightarrow 6-7t = 7 \Rightarrow t = -\frac{1}{7}$. Point $= \left(\frac{13}{7}, \frac{9}{7}, -\frac{9}{7}\right)$. (b) $\mathbf{d}\cdot\mathbf{n} = 1-4-4 = -7$; $|\mathbf{d}| = 3$, $|\mathbf{n}| = 3$. $\sin\theta = \dfrac{7}{9} \Rightarrow \theta \approx 51.1°$.

10. (a) $\overrightarrow{AB} = (k-2,2,1)$, $\overrightarrow{AC} = (6,4,2)$. (b) Since $\overrightarrow{AC} = 2\overrightarrow{AB}$ (comparing the y- and z-components: $4 = 2(2)$, $2 = 2(1)$), the x-component requires $6 = 2(k-2) \Rightarrow k = 5$. (c) Direction vector $(3,2,1)$ (from $\overrightarrow{AB}$ with $k=5$); $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. (d) Direction of L_2 is $(1,-1,1)$, not a scalar multiple of $(3,2,1)$, so L_1, L_2 are not parallel. Setting $L_1 = L_2$ (using point $(5,2,3)$ on L_2): $2+3s = 5+t$, $1+2s = 2-t$, $-1+s = 3+t$. From the first and third equations: $3s-t = 3$ and $s-t = 4 \Rightarrow t = s-4$. Substituting into $3s-t = 3$: $3s-(s-4) = 3 \Rightarrow 2s = -1 \Rightarrow s = -\frac{1}{2}, t = -\frac{9}{2}$. Checking the second equation: $1+2(-\frac{1}{2}) = 0$ but $2-(-\frac{9}{2}) = \frac{13}{2}$; since $0 \neq \frac{13}{2}$, there is no consistent solution. As L_1, L_2 are neither parallel nor intersecting, they are skew. ■