Trigonometric Functions

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Solve the equation $2\sin x = \sqrt{2}$, for $0 \le x \le 2\pi$. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Prove the identity $\dfrac{1-\sin^2 x}{1-\cos^2 x} = \cot^2 x$, for $\sin x, \cos x \neq 0$. **[4]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Consider $f(x) = 2\sin(4x) + 3$.

(a) State the amplitude of f. **[1]**

(b) State the period of f. **[2]**

(c) State the maximum and minimum values of f. **[2]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Solve the equation $\cos(2x) = -0.5$, for $0 \le x \le \pi$. **[5]**

5. [Maximum mark: 6] *Difficulty:* 2/5

The graph of $g(x) = 4\sin\left(x - \dfrac{\pi}{6}\right) - 1$ is obtained from the graph of $f(x) = \sin x$ by a sequence of transformations.

(a) Describe these transformations. **[3]**

(b) State the amplitude and period of g. **[3]**

6. [Maximum mark: 8] *Difficulty:* 3/5

Let $f(x) = \cos(x - k)$, where $0 \le x \le a$, $a, k \in \mathbb{R}^+$.

(a) Consider the case $k = \dfrac{\pi}{3}$. By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists. **[3]**

(b) Find the largest value of a for which f^{-1} exists in the case $k = \pi$. **[2]**

(c) Find the largest value of a for which f^{-1} exists in the case $\pi < k < 2\pi$. Give your answer in terms of k. **[3]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Solve the equation $2\sin^2 x - 3\cos x = 0$, for $0 \leq x \leq 2\pi$.

(a) Rewrite the equation as a quadratic in $\cos x$. **[3]**

(b) Hence solve the equation. **[4]**

8. [Maximum mark: 8] *Difficulty:* 4/5

The function f is defined by $f(x) = 3\cos^2 x - \sin^2 x$, for $0 \leq x \leq \pi$.

(a) Find the roots of the equation $f(x) = 0$. **[3]**

(b) **(i)** Find $f'(x)$. **[1]**

(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. **[4]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Using the function f from Question 8, let $g(x) = |f(x)|$, for $0 \leq x \leq \pi$.

(a) State the coordinates of the points where the graph of $y = g(x)$ meets the coordinate axes. **[3]**

(b) Hence, or otherwise, solve the inequality $|f(x)| > 1$. **[5]**

10. [Maximum mark: 16] *Difficulty:* 5/5

Challenge question. A researcher studies the growth of two bamboo shoots, P and Q, over a ten-day period. The height of shoot P, h_P cm, at time t days can be modelled by

$$h_P(t) = \sin(2t+3) + 9t + 17, \qquad 0 \leq t \leq 10.$$

The height of shoot Q, h_Q cm, at time t days can be modelled by

$$h_Q(t) = 8t + 21, \qquad 0 \leq t \leq 10.$$

(a) **(i)** Find the initial height of shoot Q. **[1]**

(ii) Find the initial height of shoot P, correct to three significant figures. **[2]**

(b) Find the value of t when $h_P(t) = h_Q(t)$, correct to three significant figures. **[3]**

(c) For $t > 5$, prove that shoot P was always taller than shoot Q. **[4]**

(d) Find the total amount of time for which the rate of growth of shoot Q was greater than the rate of growth of shoot P. **[6]**

Answer Key

1. $\sin x = \dfrac{\sqrt{2}}{2} \Rightarrow x = \dfrac{\pi}{4}$ or $x = \dfrac{3\pi}{4}$.

2. $\dfrac{1-\sin^2 x}{1-\cos^2 x} = \dfrac{\cos^2 x}{\sin^2 x} = \cot^2 x$, for $\sin x, \cos x \neq 0$. $\blacksquare$

3. (a) Amplitude $= 2$. (b) Period $= \dfrac{2\pi}{4} = \dfrac{\pi}{2}$. (c) Max $= 2+3 = 5$; Min $= -2+3 = 1$.

4. $2x = \dfrac{2\pi}{3}$ or $\dfrac{4\pi}{3}$ (within $0 \leq 2x \leq 2\pi$), so $x = \dfrac{\pi}{3}$ or $x = \dfrac{2\pi}{3}$.

5. (a) Horizontal translation of $\dfrac{\pi}{6}$ to the right, vertical stretch with scale factor 4, and vertical translation of 1 downward. (b) Amplitude $= 4$; period $= 2\pi$.

6. (a) $f(x) = \cos\left(x - \frac{\pi}{3}\right)$ is increasing on $\left[0, \frac{\pi}{3}\right]$ then decreasing, so the largest $a = \dfrac{\pi}{3}$. (b) $f(x) = \cos(x - \pi) = -\cos x$ is increasing on $[0,\pi]$ then decreasing, so the largest $a = \pi$. (c) For $\pi < k < 2\pi$, f is decreasing on $[0, k-\pi]$ then increasing, so the largest $a = k - \pi$.

7. (a) Using $\sin^2 x = 1 - \cos^2 x$: $2(1-\cos^2 x) - 3\cos x = 0 \Rightarrow 2\cos^2 x + 3\cos x - 2 = 0$. (b) $(2\cos x - 1)(\cos x + 2) = 0 \Rightarrow \cos x = \frac{1}{2}$ (rejecting $\cos x = -2$, out of range). This gives $x = \dfrac{\pi}{3}$ or $x = \dfrac{5\pi}{3}$.

8. (a) $3\cos^2 x = \sin^2 x \Rightarrow \tan^2 x = 3 \Rightarrow \tan x = \pm\sqrt{3} \Rightarrow x = \dfrac{\pi}{3}$ or $x = \dfrac{2\pi}{3}$. (b)(i) $f'(x) = -6\cos x \sin x - 2\sin x \cos x = -8\sin x \cos x = -4\sin(2x)$. (ii) $-4\sin(2x) = 0 \Rightarrow x = 0, \dfrac{\pi}{2}, \pi$. $f(0) = 3$, $f\left(\frac{\pi}{2}\right) = -1$, $f(\pi) = 3$. Points: $(0,3)$, $\left(\frac{\pi}{2}, -1\right)$, $(\pi, 3)$.

9. (a) x-intercepts at $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$ (from Question 8); y-intercept: $|f(0)| = 3$, point $(0,3)$. (b) Solving $f(x) = 1$: $3\cos^2 x - \sin^2 x = 1 \Rightarrow 3\cos^2 x - (1-\cos^2 x) = 1 \Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. Tracking the sign of f across $[0,\pi]$ (positive near the endpoints, dipping to a minimum of -1 in the middle), $|f(x)| > 1$ for $0 \leq x < \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4} < x \leq \pi$.

10. (a)(i) $h_Q(0) = 21$ cm. (ii) $h_P(0) = \sin(3) + 17 \approx 17.1$ cm. (b) Solving numerically (GDC), $h_P(t) = h_Q(t)$ at $t \approx 4.51$ days. (c) $h_P(t) - h_Q(t) = \sin(2t+3) + t - 4$. Since $\sin(2t+3) \geq -1$ always, $h_P(t) - h_Q(t) \geq t - 5$. For $t > 5$, $t - 5 > 0$, so $h_P(t) - h_Q(t) > 0$, i.e. shoot P is always taller than shoot Q. $\blacksquare$ (d) h_Q grows faster than h_P when $8 > 2\cos(2t+3) + 9 \Leftrightarrow \cos(2t+3) < -\frac{1}{2}$. Solving this inequality numerically (GDC) over $0 \leq t \leq 10$ and summing the durations gives a total time of approximately 3.72 days.