

Trigonometric Functions

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

1. [Maximum mark: 3]

Difficulty: 1/5

Solve the equation $2\sin x = 1$, for $0 \leq x \leq 2\pi$.

[3]

2. [Maximum mark: 4]

Difficulty: 1/5

Prove the identity $\frac{1 - \cos^2 x}{\sin x} = \sin x$, for $\sin x \neq 0$.

[4]

3. [Maximum mark: 5]

Difficulty: 2/5

Consider $f(x) = 3\sin(2x) + 1$.

(a) State the amplitude of f . [1]

(b) State the period of f . [2]

(c) State the maximum and minimum values of f . [2]

4. [Maximum mark: 5]

Difficulty: 2/5

Solve the equation $\cos(2x) = 0.5$, for $0 \leq x \leq \pi$.

[5]

5. [Maximum mark: 6]

Difficulty: 2/5

The graph of $g(x) = 2\sin\left(x - \frac{\pi}{3}\right) + 1$ is obtained from the graph of $f(x) = \sin x$ by a sequence of transformations.

(a) Describe these transformations. [3]

(b) State the amplitude and period of g . [3]

6. [Maximum mark: 8]

Difficulty: 3/5

Let $f(x) = \cos(x - k)$, where $0 \leq x \leq a$, $a, k \in \mathbb{R}^+$.

(a) Consider the case $k = \frac{\pi}{4}$. By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists. [3]

(b) Find the largest value of a for which f^{-1} exists in the case $k = \pi$. [2]

- (c) Find the largest value of a for which f^{-1} exists in the case $\pi < k < 2\pi$. Give your answer in terms of k . [3]

7. [Maximum mark: 7]

Difficulty: 3/5

Solve the equation $2\cos^2 x - \sin x - 1 = 0$, for $0 \leq x \leq 2\pi$.

- (a) Rewrite the equation as a quadratic in $\sin x$. [3]
(b) Hence solve the equation. [4]

8. [Maximum mark: 8]

Difficulty: 4/5

The function f is defined by $f(x) = \sin^2 x - 3\cos^2 x$, for $0 \leq x \leq \pi$.

- (a) Find the roots of the equation $f(x) = 0$. [3]
(b) (i) Find $f'(x)$. [1]
(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. [4]

9. [Maximum mark: 8]

Difficulty: 4/5

Using the function f from Question 8, let $g(x) = |f(x)|$, for $0 \leq x \leq \pi$.

- (a) State the coordinates of the points where the graph of $y = g(x)$ meets the coordinate axes. [3]
(b) Hence, or otherwise, solve the inequality $|f(x)| > 1$. [5]

10. [Maximum mark: 16]

Difficulty: 5/5

Challenge question. A researcher studies the growth of two bamboo shoots, P and Q , over a ten-day period. The height of shoot P , h_P cm, at time t days can be modelled by

$$h_P(t) = \sin(2t + 4) + 7t + 21, \quad 0 \leq t \leq 10.$$

The height of shoot Q , h_Q cm, at time t days can be modelled by

$$h_Q(t) = 6t + 25, \quad 0 \leq t \leq 10.$$

- (a) (i) Find the initial height of shoot Q . [1]
(ii) Find the initial height of shoot P , correct to three significant figures. [2]
(b) Find the value of t when $h_P(t) = h_Q(t)$, correct to three significant figures. [3]
(c) For $t > 5$, prove that shoot P was always taller than shoot Q . [4]
(d) Find the total amount of time for which the rate of growth of shoot Q was greater than the rate of growth of shoot P . [6]

Answer Key

1. $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

2. $1 - \cos^2 x = \sin^2 x$ (Pythagorean identity), so $\frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \sin x$ for $\sin x \neq 0$. ■

3. (a) Amplitude = 3. (b) Period = $\frac{2\pi}{2} = \pi$. (c) Max = $3 + 1 = 4$; Min = $-3 + 1 = -2$.

4. $2x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ (within $0 \leq 2x \leq 2\pi$), so $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

5. (a) Horizontal translation of $\frac{\pi}{3}$ to the right, vertical stretch with scale factor 2, and vertical translation of 1 upward. (b) Amplitude = 2; period = 2π .

6. (a) $f(x) = \cos(x - \frac{\pi}{4})$ is increasing on $[0, \frac{\pi}{4}]$ then decreasing, so the largest $a = \frac{\pi}{4}$. (b) $f(x) = \cos(x - \pi) = -\cos x$ is increasing on $[0, \pi]$ then decreasing, so the largest $a = \pi$. (c) For $\pi < k < 2\pi$, f is decreasing on $[0, k - \pi]$ then increasing, so the largest $a = k - \pi$.

7. (a) Using $\cos^2 x = 1 - \sin^2 x$: $2(1 - \sin^2 x) - \sin x - 1 = 0 \Rightarrow 2\sin^2 x + \sin x - 1 = 0$. (b) $(2\sin x - 1)(\sin x + 1) = 0 \Rightarrow \sin x = \frac{1}{2}$ or $\sin x = -1$. This gives $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

8. (a) $\sin^2 x = 3\cos^2 x \Rightarrow \tan^2 x = 3 \Rightarrow \tan x = \pm\sqrt{3} \Rightarrow x = \frac{\pi}{3}$ or $x = \frac{2\pi}{3}$. (b)(i) $f'(x) = 2\sin x \cos x + 6\sin x \cos x = 8\sin x \cos x = 4\sin(2x)$. (ii) $4\sin(2x) = 0 \Rightarrow x = 0, \frac{\pi}{2}, \pi$. $f(0) = -3$, $f(\frac{\pi}{2}) = 1$, $f(\pi) = -3$. Points: $(0, -3)$, $(\frac{\pi}{2}, 1)$, $(\pi, -3)$.

9. (a) x -intercepts at $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$ (from Question 8); y -intercept: $|f(0)| = 3$, point $(0, 3)$. (b) Solving $f(x) = -1$: $\sin^2 x - 3\cos^2 x = -1 \Rightarrow (1 - \cos^2 x) - 3\cos^2 x = -1 \Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. Tracking the sign of f across $[0, \pi]$ (negative near the endpoints, positive with maximum 1 in the middle), $|f(x)| > 1$ for $0 \leq x < \frac{\pi}{4}$ or $\frac{3\pi}{4} < x \leq \pi$.

10. (a)(i) $h_Q(0) = 25$ cm. (ii) $h_P(0) = \sin(4) + 21 \approx 20.2$ cm. (b) Solving numerically (GDC), $h_P(t) = h_Q(t)$ at $t \approx 4.19$ days. (c) $h_P(t) - h_Q(t) = \sin(2t + 4) + t - 4$. Since $\sin(2t + 4) \geq -1$ always, $h_P(t) - h_Q(t) \geq t - 5$. For $t > 5$, $t - 5 > 0$, so $h_P(t) - h_Q(t) > 0$, i.e. shoot P is always taller than shoot Q . ■ (d) h_Q grows faster than h_P when $6 > 2\cos(2t + 4) + 7 \Leftrightarrow \cos(2t + 4) < -\frac{1}{2}$. Solving this inequality numerically (GDC) over $0 \leq t \leq 10$ and summing the durations gives a total time of approximately 3.24 days.