Sequences and Series

Worksheet 3 — Advanced

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 6] *Difficulty:* 3/5

Consider the series $\log_2 x + \log_2 x^{1/2} + \log_2 x^{1/4} + \dots$, where $x \in \mathbb{R}$, $x > 1$.

(a) Show that the series is geometric with common ratio $\frac{1}{2}$. **[3]**

(b) Show that $S_\infty = 2\log_2 x$. **[3]**

2. [Maximum mark: 6] *Difficulty:* 3/5

An arithmetic series has first term $\ln 2$ and common difference $\ln 3$.

(a) Find S_n, the sum of the first n terms, in terms of n. **[3]**

(b) Show that S_n can be written as $\ln\left(2^n\, 3^{n(n-1)/2}\right)$. **[3]**

3. [Maximum mark: 7] *Difficulty:* 3/5

(a) Starting from $S_n = \dfrac{a(1-r^n)}{1-r}$ for $|r| < 1$, and using the fact that $r^n \to 0$ as $n \to \infty$, derive the formula $S_\infty = \dfrac{a}{1-r}$. **[4]**

(b) Hence evaluate $\displaystyle\sum_{n=0}^{\infty} 3(0.25)^n$. **[3]**

4. [Maximum mark: 8] *Difficulty:* 4/5

Let $\theta \neq 2k\pi$ for $k \in \mathbb{Z}$.

(a) By summing a geometric series, show that

$$\sum_{k=0}^{n-1} e^{ik\theta} = \frac{e^{in\theta} - 1}{e^{i\theta} - 1}.$$

[3]

(b) By writing $e^{i\theta} - 1 = 2i\,e^{i\theta/2}\sin\left(\frac{\theta}{2}\right)$ and applying the same identity to the numerator, show that

$$\sum_{k=0}^{n-1} e^{ik\theta} = e^{i(n-1)\theta/2} \cdot \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

[4]

(c) Hence write down an expression for $\sum_{k=0}^{n-1} \cos(k\theta)$. **[1]**

5. [Maximum mark: 8] *Difficulty: 4/5*

(a) By differentiating both sides of $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$ $(x \neq 1)$ with respect to x, show that

$$\sum_{k=1}^{n} k\,x^{k-1} = \frac{n\,x^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

[5]

(b) Hence find $\sum_{k=1}^{5} k(2)^{k-1}$ using this formula, and verify your answer by direct computation. **[3]**

6. [Maximum mark: 9] *Difficulty: 4/5*

A sequence is defined by $u_1 = 1$ and $u_{n+1} = u_n + n$ for $n \in \mathbb{Z}^+$.

(a) Verify that $u_n = 1 + \frac{n(n-1)}{2}$ holds for $n = 1, 2, 3$. **[2]**

(b) Prove by mathematical induction that $u_n = 1 + \frac{n(n-1)}{2}$ for all $n \in \mathbb{Z}^+$. **[7]**

7. [Maximum mark: 9] *Difficulty: 4/5*

Determine whether each of the following infinite series converges. If it converges, find its sum; if it diverges, state this and give a brief reason.

(a) $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ **[3]**

(b) $\sum_{n=1}^{\infty} n$ **[3]**

(c) $\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$ **[3]**

8. [Maximum mark: 11] *Difficulty: 5/5*

Consider the series $\ln x + m\ln x + \frac{1}{4}\ln x + \ldots$, where $x \in \mathbb{R}$, $x > 1$ and $m \in \mathbb{R}$, $m \neq 0$.

(a) Given that the series is geometric, show that $m = \pm\frac{1}{2}$. **[3]**

(b) Given further that $m > 0$, find the sum to infinity of the series, in terms of $\ln x$. **[3]**

(c) Given instead that the series is arithmetic, show that $m = \frac{5}{8}$. **[3]**

(d) For the arithmetic case in part (c), find S_5, the sum of the first five terms, in terms of $\ln x$. **[2]**

9. [Maximum mark: 10] *Difficulty: 5/5*

Let a, b, c be three real numbers, no two of which are equal, which are simultaneously three consecutive terms of an arithmetic sequence and three consecutive terms of a geometric sequence.

(a) Using the arithmetic condition, express c in terms of a and b. **[2]**

(b) Using the geometric condition and your result from part (a), show that $(b-a)^2 = 0$. **[5]**

(c) Hence prove that $a = b = c$, and explain why this contradicts the requirement that no two of a, b, c are equal. **[3]**

10. [Maximum mark: 14] *Difficulty: 5/5*

Challenge question. A geometric series has positive first term a and common ratio r, where $0 < r < 1$. Let S denote the (fixed, positive) sum to infinity of the series.

(a) Express a in terms of S and r. **[2]**

(b) Show that $u_1u_2 = S^2r(1-r)^2$. **[2]**

(c) By differentiating with respect to r, show that u_1u_2 is maximised when $r = \dfrac{1}{3}$, and find this maximum value in terms of S. **[8]**

(d) State, with a reason, whether this maximum value of u_1u_2 could ever be equal to S^2. **[2]**

Answer Key

1. (a) The kth term is $\log_2 x^{(1/2)^{k-1}} = \left(\frac{1}{2}\right)^{k-1} \log_2 x$, so consecutive terms have ratio $\frac{1}{2}$. (b) $S_\infty = \dfrac{\log_2 x}{1-\frac{1}{2}} = 2\log_2 x$.

2. (a) $S_n = \dfrac{n}{2}\left(2\ln 2 + (n-1)\ln 3\right) = n\ln 2 + \dfrac{n(n-1)}{2}\ln 3$. (b) $S_n = \ln(2^n) + \ln\left(3^{n(n-1)/2}\right) = \ln\left(2^n 3^{n(n-1)/2}\right)$.

3. (a) As $n \to \infty$, $r^n \to 0$, so $S_n = \dfrac{a(1-r^n)}{1-r} \to \dfrac{a(1-0)}{1-r} = \dfrac{a}{1-r} = S_\infty$. (b) $a = 3$, $r = 0.25$: $S_\infty = \dfrac{3}{0.75} = 4$.

4. (a) This is a geometric series with first term 1 and ratio $e^{i\theta}$, so $\sum_{k=0}^{n-1} e^{ik\theta} = \dfrac{(e^{i\theta})^n - 1}{e^{i\theta}-1} = \dfrac{e^{in\theta}-1}{e^{i\theta}-1}$. (b) Writing both numerator and denominator in the given form: $e^{in\theta} - 1 = 2i\,e^{in\theta/2}\sin\left(\frac{n\theta}{2}\right)$ and $e^{i\theta} - 1 = 2i\,e^{i\theta/2}\sin\left(\frac{\theta}{2}\right)$. Dividing gives $e^{i(n-1)\theta/2}\dfrac{\sin(n\theta/2)}{\sin(\theta/2)}$. (c) Taking the real part: $\displaystyle\sum_{k=0}^{n-1}\cos(k\theta) = \cos\left(\frac{(n-1)\theta}{2}\right)\frac{\sin(n\theta/2)}{\sin(\theta/2)}$.

5. (a) Differentiating $\dfrac{x^{n+1}-1}{x-1}$ using the quotient rule gives $\dfrac{(n+1)x^n(x-1)-(x^{n+1}-1)}{(x-1)^2} = \dfrac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$, which equals $\sum_{k=1}^{n} kx^{k-1}$ termwise. (b) With $n = 5$, $x = 2$: numerator $= 5(2)^6 - 6(2)^5 + 1 = 320 - 192 + 1 = 129$, denominator $= 1$, so the sum is 129. Direct check: $1 + 4 + 12 + 32 + 80 = 129$.

6. (a) $n = 1$: $1 + 0 = 1 = u_1$. $n = 2$: $1 + 1 = 2 = u_2$ (since $u_2 = u_1 + 1 = 2$). $n = 3$: $1 + 3 = 4 = u_3$ (since $u_3 = u_2 + 2 = 4$). (b) *Base case* $n = 1$: shown above. *Inductive step*: assume $u_k = 1 + \dfrac{k(k-1)}{2}$. Then $u_{k+1} = u_k + k = 1 + \dfrac{k(k-1)}{2} + k = 1 + \dfrac{k^2-k+2k}{2} = 1 + \dfrac{k(k+1)}{2}$, which is the required form with $n = k+1$. Since true for $n = 1$ and true for $n = k+1$ whenever true for $n = k$, by induction it holds for all $n \in \mathbb{Z}^+$.

7. (a) Converges (geometric, $|r| = \frac{3}{4} < 1$): $S_\infty = \dfrac{3/4}{1-3/4} = 3$. (b) Diverges: partial sums increase without bound. (c) Converges (geometric, $|r| = \frac{2}{3} < 1$): $S_\infty = \dfrac{-2/3}{1+2/3} = -\dfrac{2}{5}$.

8. (a) Geometric requires $\dfrac{m\ln x}{\ln x} = \dfrac{\frac{1}{4}\ln x}{m\ln x} \Rightarrow m = \dfrac{1}{4m} \Rightarrow m^2 = \dfrac{1}{4} \Rightarrow m = \pm\dfrac{1}{2}$. (b) $m = \frac{1}{2} \Rightarrow r = \frac{1}{2}$; $S_\infty = \dfrac{\ln x}{1-1/2} = 2\ln x$. (c) Arithmetic requires $(m-1)\ln x = \left(\frac{1}{4} - m\right)\ln x \Rightarrow 2m = \frac{5}{4} \Rightarrow m = \frac{5}{8}$. (d) $d = \left(\frac{5}{8} - 1\right)\ln x = -\frac{3}{8}\ln x$; $S_5 = \dfrac{5}{2}\left(2\ln x + 4(-\frac{3}{8}\ln x)\right) = \dfrac{5}{2}\left(2\ln x - \frac{3}{2}\ln x\right) = \dfrac{5}{2}\cdot\dfrac{1}{2}\ln x = \dfrac{5}{4}\ln x$.

9. (a) From $2b = a + c$ (AP condition), $c = 2b - a$. (b) GP condition: $b^2 = ac = a(2b-a) = 2ab - a^2 \Rightarrow b^2 - 2ab + a^2 = 0 \Rightarrow (b-a)^2 = 0$. (c) $(b-a)^2 = 0 \Rightarrow a = b$; then $c = 2b - a = 2a - a = a$, so $a = b = c$. This contradicts the assumption that no two of a, b, c are equal — so in fact no such sequence with distinct terms exists.

10. (a) $S = \dfrac{a}{1-r} \Rightarrow a = S(1-r)$. (b) $u_1u_2 = a \cdot ar = a^2r = S^2(1-r)^2r$. (c) $P(r) = S^2r(1-r)^2$. $P'(r) = S^2\left[(1-r)^2 - 2r(1-r)\right] = S^2(1-r)(1-3r)$. Setting $P'(r) = 0$ for $0 < r < 1$ gives $r = \frac{1}{3}$ (since $r = 1$ is excluded). $P'(r) > 0$ for $r < \frac{1}{3}$ and $P'(r) < 0$ for $r > \frac{1}{3}$, confirming a maximum at $r = \frac{1}{3}$. Maximum value: $P\left(\frac{1}{3}\right) = S^2 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2 = \dfrac{4S^2}{27}$. (d) No: $\dfrac{4S^2}{27} = S^2$ would require $\dfrac{4}{27} = 1$, which is false for any $S \neq 0$. Since $S > 0$, the maximum value of u_1u_2 is always strictly less than S^2.