Sequences and Series

Worksheet 2 — Intermediate

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 5] *Difficulty: 2/5*

Three consecutive terms of an arithmetic sequence are $x-3,\ x+1,\ 2x+4$.

(a) Find the value of x. **[3]**

(b) Hence find the three terms and the common difference. **[2]**

2. [Maximum mark: 6] *Difficulty: 2/5*

Three consecutive terms of a geometric sequence are $x-1,\ x+1,\ 3x-1$.

(a) Show that $2x(x-3)=0$. **[3]**

(b) Hence find the two possible values of x, and the corresponding common ratio r in each case. **[3]**

3. [Maximum mark: 6] *Difficulty: 3/5*

The sum of the first n terms of a sequence is given by $S_n = 3n^2 - n$.

(a) Find u_1. **[2]**

(b) Find u_n in terms of n, and hence show that the sequence is arithmetic, stating the common difference. **[4]**

4. [Maximum mark: 7] *Difficulty: 3/5*

Maria invests \$2000 in an account that pays 4% interest compounded annually. Let u_n be the value of the investment after n complete years.

(a) Show that $u_n = 2000(1.04)^n$. **[2]**

(b) Find the value of the investment after 10 years, to the nearest dollar. **[2]**

(c) Find the least value of n for which the investment exceeds \$3000. **[3]**

5. [Maximum mark: 7] *Difficulty: 3/5*

A geometric series has first term $\sin^2\theta$ and common ratio $\cos^2\theta$, where $0<\theta<\dfrac{\pi}{2}$.

(a) State why this series converges for all θ in the given domain. **[2]**

(b) Show that $S_\infty = 1$. **[3]**

(c) Find an expression for S_3, the sum of the first three terms, in its simplest form. **[2]**

6. [Maximum mark: 8] *Difficulty:* 4/5

A sequence is defined by $u_1 = 3$ and $u_{n+1} = 2u_n - 1$ for $n \in \mathbb{Z}^+$.

(a) Let $v_n = u_n - 1$. Show that $v_{n+1} = 2v_n$. **[3]**

(b) Hence find an expression for u_n in terms of n. **[2]**

(c) Find $\displaystyle\sum_{n=1}^{10} u_n$. **[3]**

7. [Maximum mark: 8] *Difficulty:* 4/5

(a) Verify that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ for $k \in \mathbb{Z}^+$. **[2]**

(b) Hence show that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$. **[4]**

(c) State the value of $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. **[2]**

8. [Maximum mark: 9] *Difficulty:* 4/5

The first, second and fourth terms of a geometric sequence with first term $a \neq 0$ and common ratio r form the first three terms of an arithmetic sequence with nonzero common difference.

(a) Show that $r^3 - 2r + 1 = 0$. **[3]**

(b) By factorising, show that $r = 1$ is a root of this equation, and find the other two roots. Explain why $r = 1$ must be rejected. **[4]**

(c) Hence state the possible value(s) of r, giving your answer(s) in exact form. **[2]**

9. [Maximum mark: 9] *Difficulty:* 5/5

Consider the geometric series with first term 1 and common ratio $z = e^{i\pi/3}$.

(a) Show that S_6, the sum of the first six terms of the series, is equal to 0. **[4]**

(b) Hence deduce the values of $\displaystyle\sum_{k=0}^{5} \cos\left(\frac{k\pi}{3}\right)$ and $\displaystyle\sum_{k=0}^{5} \sin\left(\frac{k\pi}{3}\right)$. **[4]**

(c) Explain briefly why this result is expected, given the geometric interpretation of z as a sixth root of unity. **[1]**

10. [Maximum mark: 12] *Difficulty:* 5/5

Challenge question. The first three terms of a geometric sequence, u_1, u_2, u_3, satisfy

$$u_1 + u_2 + u_3 = 26 \qquad \text{and} \qquad u_1u_2u_3 = 216.$$

(a) By using the property $u_1u_3 = u_2^2$, show that $u_2 = 6$. **[3]**

(b) Find the two possible geometric sequences satisfying both conditions, and state the common ratio r in each case. **[6]**

(c) Determine which of the two sequences found in part (b) has a convergent sum to infinity, and find its value. **[3]**

Answer Key

1. (a) $2(x+1) = (x-3)+(2x+4) \Rightarrow 2x+2 = 3x+1 \Rightarrow x = 1$. (b) Terms: $-2, 2, 6$; common difference $d = 4$.

2. (a) $(x+1)^2 = (x-1)(3x-1) \Rightarrow x^2+2x+1 = 3x^2-4x+1 \Rightarrow 2x^2-6x = 0 \Rightarrow 2x(x-3) = 0$. (b) $x = 0$: terms $-1, 1, -1$, $r = -1$. $x = 3$: terms $2, 4, 8$, $r = 2$.

3. (a) $u_1 = S_1 = 3(1)-1 = 2$. (b) $u_n = S_n - S_{n-1} = 3n^2-n-[3(n-1)^2-(n-1)] = 6n-4$. Since $u_n - u_{n-1} = 6$ (constant), the sequence is arithmetic with $d = 6$.

4. (a) Each year the value is multiplied by 1.04, so after n years $u_n = 2000(1.04)^n$. (b) $u_{10} = 2000(1.04)^{10} \approx$ \$2960. (c) $2000(1.04)^n > 3000 \Rightarrow 1.04^n > 1.5 \Rightarrow n > \dfrac{\ln 1.5}{\ln 1.04} \approx 10.3$, so least $n = 11$.

5. (a) $|\cos^2\theta| < 1$ for $0 < \theta < \pi/2$, so the common ratio satisfies $|r| < 1$ and the series converges. (b) $S_\infty = \dfrac{\sin^2\theta}{1-\cos^2\theta} = \dfrac{\sin^2\theta}{\sin^2\theta} = 1$. (c) $S_3 = \dfrac{\sin^2\theta(1-\cos^6\theta)}{1-\cos^2\theta} = 1-\cos^6\theta$.

6. (a) $v_{n+1} = u_{n+1}-1 = 2u_n-2 = 2(u_n-1) = 2v_n$. (b) $v_1 = 2$, so $v_n = 2^n$ (geometric, $r = 2$), giving $u_n = 2^n+1$. (c) $\displaystyle\sum_{n=1}^{10} u_n = \sum_{n=1}^{10} 2^n + 10 = (2^{11}-2)+10 = 2056$.

7. (a) $\dfrac{1}{k} - \dfrac{1}{k+1} = \dfrac{(k+1)-k}{k(k+1)} = \dfrac{1}{k(k+1)}$. (b) The sum telescopes: $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1-\frac{1}{n+1} = \frac{n}{n+1}$. (c) As $n \to \infty$, $\dfrac{n}{n+1} \to 1$, so the infinite sum equals 1.

8. (a) $r = ar/a$, and the AP condition gives $2(ar) = a+ar^3 \Rightarrow 2r = 1+r^3 \Rightarrow r^3-2r+1 = 0$. (b) $r = 1$: $1-2+1 = 0$, confirming $r = 1$ is a root. Factorising: $r^3-2r+1 = (r-1)(r^2+r-1)$, so the other roots satisfy $r^2+r-1 = 0 \Rightarrow r = \dfrac{-1\pm\sqrt{5}}{2}$. $r = 1$ gives common difference 0, which is excluded. (c) $r = \dfrac{-1+\sqrt{5}}{2}$ or $r = \dfrac{-1-\sqrt{5}}{2}$.

9. (a) Since $z^6 = e^{i2\pi} = 1$ and $z \neq 1$, $S_6 = \dfrac{z^6-1}{z-1} = \dfrac{0}{z-1} = 0$. (b) Taking real and imaginary parts of $S_6 = 0$: $\displaystyle\sum_{k=0}^{5}\cos\left(\frac{k\pi}{3}\right) = 0$ and $\displaystyle\sum_{k=0}^{5}\sin\left(\frac{k\pi}{3}\right) = 0$. (c) The six values of z^k are precisely the six sixth roots of unity, symmetrically placed around the unit circle, so their (vector) sum is zero.

10. (a) For a GP, $u_1u_3 = u_2^2$, so $u_1u_2u_3 = u_2^3 = 216 \Rightarrow u_2 = 6$. (b) Then $u_1+u_3 = 20$ and $u_1u_3 = 36$, so u_1, u_3 are roots of $t^2-20t+36 = 0 \Rightarrow t = 18$ or $t = 2$. This gives sequence $2, 6, 18, \ldots$ with $r = 3$, or sequence $18, 6, 2, \ldots$ with $r = \frac{1}{3}$. (c) Only $|r| < 1$ gives convergence, so the sequence $18, 6, 2, \ldots$ ($r = \frac{1}{3}$) converges: $S_\infty = \dfrac{18}{1-\frac{1}{3}} = 27$.