Sequences and Series

Worksheet 1 — Foundations

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

An arithmetic sequence has first term $u_1 = 7$ and common difference $d = 3$.

(a) Find u_{10}. **[2]**

(b) Find S_{10}, the sum of the first 10 terms. **[2]**

2. [Maximum mark: 5] *Difficulty:* 1/5

A geometric sequence has first term $u_1 = 5$ and common ratio $r = 2$.

(a) Find u_6. **[2]**

(b) Find S_6, the sum of the first 6 terms. **[3]**

3. [Maximum mark: 5] *Difficulty:* 2/5

The nth term of an arithmetic sequence is given by $u_n = 25 - 4n$.

(a) State the value of the first term, u_1. **[1]**

(b) Given that the nth term of this sequence is -19, find the value of n. **[2]**

(c) Find the common difference, d. **[2]**

4. [Maximum mark: 5] *Difficulty:* 2/5

In a geometric sequence, $u_2 = 6$ and $u_5 = 48$.

(a) Find the common ratio, r. **[2]**

(b) Find the first term, u_1. **[1]**

(c) Find S_{10}. **[2]**

5. [Maximum mark: 6] *Difficulty:* 2/5

The sum of the first n terms of an arithmetic series is given by $S_n = n^2 + 3n$.

(a) Find u_1. **[2]**

(b) Find the common difference, d. **[2]**

(c) Hence write down an expression for u_n in terms of n. **[2]**

6. [Maximum mark: 6] *Difficulty: 3/5*

An infinite geometric series has first term 12 and sum to infinity 20.

(a) Find the common ratio, r. **[2]**

(b) Find the least value of n such that $S_n > 19.5$. **[4]**

7. [Maximum mark: 7] *Difficulty: 3/5*

Consider the sum $\sum_{k=1}^{20}(3k-2)$.

(a) Write down the first three terms of the series, and state the common difference. **[2]**

(b) Evaluate $\sum_{k=1}^{20}(3k-2)$ using an appropriate formula. **[3]**

(c) Show that, in general, $\sum_{k=1}^{n}(3k-2) = \dfrac{n(3n-1)}{2}$. **[2]**

8. [Maximum mark: 7] *Difficulty: 4/5*

A ball is dropped from a height of 4 metres. Each time it hits the ground it bounces back up to $\frac{3}{4}$ of the height it fell from, so that the successive bounce heights form a geometric sequence.

(a) Find the height reached after the third bounce. **[2]**

(b) Assuming the ball continues to bounce indefinitely, find the total vertical distance travelled by the ball. **[5]**

9. [Maximum mark: 8] *Difficulty: 4/5*

(a) Using the formula for the sum of an arithmetic series, prove that

$$2+4+6+\cdots+2n = n(n+1)$$

for all $n \in \mathbb{Z}^+$. **[4]**

(b) Hence, or otherwise, show that the sum of the first n positive odd integers is n^2. **[4]**

10. [Maximum mark: 10] *Difficulty: 5/5*

Challenge question. A sequence is defined by $a_1 = 2$ and

$$a_{n+1} = a_n^2 - a_n + 1, \qquad n \in \mathbb{Z}^+.$$

(a) Find a_2, a_3 and a_4. **[2]**

(b) Prove by mathematical induction that

$$\sum_{i=1}^{n} \frac{1}{a_i} = 1 - \frac{1}{a_{n+1}-1}$$

for all $n \in \mathbb{Z}^+$. **[8]**

Answer Key

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1. (a) $u_{10} = 7 + 9(3) = 34$. (b) $S_{10} = \frac{10}{2}\big(2(7) + 9(3)\big) = 5(41) = 205$.

2. (a) $u_6 = 5(2)^5 = 160$. (b) $S_6 = \frac{5(2^6 - 1)}{2 - 1} = 5(63) = 315$.

3. (a) $u_1 = 21$. (b) $25 - 4n = -19 \Rightarrow n = 11$. (c) $d = -4$.

4. (a) $u_1 r^3 = 48/6 = 8 \Rightarrow r = 2$. (b) $u_1 = 6/2 = 3$. (c) $S_{10} = \frac{3(2^{10} - 1)}{1} = 3069$.

5. (a) $u_1 = S_1 = 4$. (b) $u_2 = S_2 - S_1 = 10 - 4 = 6 \Rightarrow d = 2$. (c) $u_n = 4 + (n-1)(2) = 2n + 2$.

6. (a) $\frac{12}{1-r} = 20 \Rightarrow 1 - r = 0.6 \Rightarrow r = 0.4$. (b) $S_n = 20(1 - 0.4^n)$. Solving $20(1 - 0.4^n) > 19.5$ gives $0.4^n < 0.025$, so $n > 4.03$. Testing $n = 4$: $S_4 = 19.488$ (fails); $n = 5$: $S_5 = 19.795$ (succeeds). Least $n = 5$.

7. (a) First three terms: $1, 4, 7$; common difference $d = 3$. (b) $S_{20} = \frac{20}{2}(2(1) + 19(3)) = 10(59) = 590$.
(c) $\sum_{k=1}^{n}(3k - 2) = 3 \cdot \frac{n(n+1)}{2} - 2n = \frac{3n^2 + 3n - 4n}{2} = \frac{n(3n-1)}{2}$, confirmed by $n = 20$ giving 590.

8. (a) Height after 3rd bounce $= 4(0.75)^3 = 1.6875$ m. (b) Total distance $= 4 + 2\sum_{k=1}^{\infty} 4(0.75)^k = 4 + 2\left(\frac{3}{1 - 0.75}\right) = 4 + 2(12) = 28$ m.

9. (a) $2 + 4 + \cdots + 2n = \frac{n}{2}(2 \cdot 2 + (n-1) \cdot 2) = n(n+1)$. (b) Sum of first $2n$ positive integers $= n(2n+1)$. Subtracting the sum of the even terms found in (a): sum of odd terms $= n(2n+1) - n(n+1) = n^2$.

10. (a) $a_2 = 4 - 2 + 1 = 3$, $a_3 = 9 - 3 + 1 = 7$, $a_4 = 49 - 7 + 1 = 43$.
(b) *Base case* ($n = 1$): LHS $= 1/2$. RHS $= 1 - \frac{1}{a_2 - 1} = 1 - \frac{1}{2} = \frac{1}{2}$. True.

Inductive step: Assume $\sum_{i=1}^{k} \frac{1}{a_i} = 1 - \frac{1}{a_{k+1} - 1}$. Then

$$\sum_{i=1}^{k+1} \frac{1}{a_i} = 1 - \frac{1}{a_{k+1} - 1} + \frac{1}{a_{k+1}}.$$

Since $a_{k+2} - 1 = a_{k+1}^2 - a_{k+1} = a_{k+1}(a_{k+1} - 1)$, partial fractions give $\frac{1}{a_{k+2} - 1} = \frac{1}{a_{k+1} - 1} - \frac{1}{a_{k+1}}$, so

$$1 - \frac{1}{a_{k+2} - 1} = 1 - \frac{1}{a_{k+1} - 1} + \frac{1}{a_{k+1}} = \sum_{i=1}^{k+1} \frac{1}{a_i}.$$

So the statement holds for $n = k + 1$ whenever it holds for $n = k$. Since it is true for $n = 1$, by mathematical induction it is true for all $n \in \mathbb{Z}^+$. ■