Rational Functions

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

State the domain of $f(x) = \dfrac{x+5}{x-6}$. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

For $f(x) = \dfrac{7x+3}{x-4}$, state the equation of

(a) the vertical asymptote; **[2]**

(b) the horizontal asymptote. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Consider $f(x) = \dfrac{5x+10}{x-3}$.

(a) Find the coordinates of the x-intercept. **[2]**

(b) Find the coordinates of the y-intercept. **[1]**

(c) State the equation of the vertical asymptote. **[2]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Express $\dfrac{1}{x-3} + \dfrac{1}{x+5}$ as a single fraction.

(a) Combine the two fractions. **[4]**

(b) State the values of x excluded from the domain. **[2]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Solve the equation $\dfrac{x+3}{x-5} = 4$.

(a) Solve the equation. **[4]**

(b) Verify that your solution satisfies the necessary restriction on x. **[2]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider $f(x) = \dfrac{6x-3}{x+4}$.

(a) State the equations of the vertical and horizontal asymptotes of $y = f(x)$. **[2]**

(b) Find the coordinates of the x-intercept and the y-intercept. **[2]**

(c) Hence solve the inequality $f(x) > 0$. **[3]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Solve the inequality $\dfrac{4x+1}{x^2-3x-10} \geq 0$.

(a) Factorise the denominator. **[2]**

(b) State the critical values of x. **[2]**

(c) Hence solve the inequality. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Express $\dfrac{9x-4}{(x-3)(x+2)}$ in the form $\dfrac{A}{x-3} + \dfrac{B}{x+2}$.

(a) Set up an equation for A and B. **[3]**

(b) Solve for A and B. **[5]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Let $g(x) = \dfrac{1}{x^2-12x+35}$, where $x \in \mathbb{R}$, $x > 7$.

(a) Show that $g^{-1}(x) = 6 + \dfrac{\sqrt{x^2+x}}{x}$. **[5]**

(b) State the domain of g^{-1}. **[3]**

10. [Maximum mark: 21] *Difficulty:* 5/5

Challenge question. Consider $f(x) = \dfrac{1}{x^2-14x+48}$, where $x \in \mathbb{R}$, $x \neq 6$, $x \neq 8$.

(a) State the equations of the vertical asymptotes and the horizontal asymptote of $y = f(x)$. **[2]**

(b) For $6 < x < 8$, find the coordinates of the local maximum or minimum point of $y = f(x)$, stating which it is. **[4]**

Let $g(x) = f(x)$, where $x \in \mathbb{R}$, $x > 8$.

(c) Show that $g^{-1}(x) = 7 + \dfrac{\sqrt{x^2+x}}{x}$. **[6]**

(d) State the domain of g^{-1}. **[2]**

Let $h(x) = \arctan\left(\dfrac{x}{8}\right)$, where $x \in \mathbb{R}$.

(e) Given that $(h \circ g)(a) = \dfrac{\pi}{4}$ for some $a > 8$, find the value of a. Give your answer in the form $p + \dfrac{q}{4}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. **[7]**

Answer Key

1. Require $x-6 \neq 0 \Rightarrow x \neq 6$.

2. (a) $x=4$. (b) $y=7$.

3. (a) $5x+10=0 \Rightarrow x=-2$; point $(-2,0)$. (b) $f(0)=\dfrac{10}{-3}=-\frac{10}{3}$; point $\left(0,-\frac{10}{3}\right)$. (c) $x=3$.

4. (a) $\dfrac{1}{x-3}+\dfrac{1}{x+5}=\dfrac{(x+5)+(x-3)}{(x-3)(x+5)}=\dfrac{2x+2}{x^2+2x-15}$. (b) $x \neq 3$ and $x \neq -5$.

5. (a) $x+3=4(x-5) \Rightarrow x+3=4x-20 \Rightarrow x=\dfrac{23}{3}$. (b) $x=\frac{23}{3} \neq 5$, so the solution is valid.

6. (a) VA: $x=-4$; HA: $y=6$. (b) x-intercept: $6x-3=0 \Rightarrow x=\frac{1}{2}$, i.e. $\left(\frac{1}{2},0\right)$. y-intercept: $f(0)=-\frac{3}{4}$, i.e. $\left(0,-\frac{3}{4}\right)$. (c) Testing each interval determined by $x=-4$ and $x=\frac{1}{2}$: $f(x)>0$ for $x<-4$ or $x>\frac{1}{2}$.

7. (a) $x^2-3x-10=(x-5)(x+2)$. (b) Critical values: $x=-\frac{1}{4}$ (numerator zero), $x=5$ and $x=-2$ (denominator zero, excluded). (c) Testing each interval: $f(x) \geq 0$ for $-2<x \leq -\frac{1}{4}$ or $x>5$.

8. (a) $9x-4=A(x+2)+B(x-3)$. (b) $x=3$: $23=5A \Rightarrow A=\frac{23}{5}$. $x=-2$: $-22=-5B \Rightarrow B=\frac{22}{5}$.

9. (a) $y=\dfrac{1}{x^2-12x+35} \Rightarrow x^2-12x+35=\dfrac{1}{y} \Rightarrow x^2-12x+\left(35-\dfrac{1}{y}\right)=0$. Using the quadratic formula: $x=6 \pm \sqrt{36-35+\frac{1}{y}}=6 \pm \sqrt{1+\frac{1}{y}}$. Since $x>7$, take the $+$ root: $x=6+\sqrt{1+\frac{1}{y}}$. Swapping x,y: $g^{-1}(x)=6+\sqrt{1+\frac{1}{x}}=6+\dfrac{\sqrt{x^2+x}}{x}$. (b) As $x \to 7^+$, $g(x) \to +\infty$; as $x \to \infty$, $g(x) \to 0^+$. Range of g is $(0,\infty)$, so domain of g^{-1} is $x>0$.

10. (a) VA: $x=6$, $x=8$; HA: $y=0$. (b) $f'(x)=\dfrac{-(2x-14)}{(x^2-14x+48)^2}=0 \Rightarrow x=7$. At $x=7$: $f(7)=\dfrac{1}{49-98+48}=\dfrac{1}{-1}=-1$. Since $f(x) \to -\infty$ as $x \to 6^+$ or $x \to 8^-$, the point $(7,-1)$ is a local *maximum*. (c) $x=7 \pm \sqrt{49-48+\frac{1}{y}}=7 \pm \sqrt{1+\frac{1}{y}}$. Since $x>8$, take the $+$ root: $g^{-1}(x)=7+\sqrt{1+\frac{1}{x}}=7+\dfrac{\sqrt{x^2+x}}{x}$. (d) Range of g (for $x>8$) is $(0,\infty)$, so domain of g^{-1} is $x>0$. (e) $(h \circ g)(a)=\dfrac{\pi}{4} \Rightarrow \dfrac{g(a)}{8}=1 \Rightarrow g(a)=8 \Rightarrow \dfrac{1}{a^2-14a+48}=8 \Rightarrow a^2-14a+48=\dfrac{1}{8} \Rightarrow a^2-14a+\dfrac{383}{8}=0$. So $a=7 \pm \sqrt{49-\frac{383}{8}}=7 \pm \sqrt{\frac{9}{8}}=7 \pm \dfrac{3\sqrt{2}}{4}$. Since $a>8$, $a=7+\dfrac{3}{4}\sqrt{2}$.