Rational Functions

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

State the domain of $f(x) = \dfrac{x-3}{x+4}$. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

For $f(x) = \dfrac{5x+2}{x-3}$, state the equation of

(a) the vertical asymptote; **[2]**

(b) the horizontal asymptote. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Consider $f(x) = \dfrac{3x+6}{x-2}$.

(a) Find the coordinates of the x-intercept. **[2]**

(b) Find the coordinates of the y-intercept. **[1]**

(c) State the equation of the vertical asymptote. **[2]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Express $\dfrac{1}{x-1} + \dfrac{1}{x+4}$ as a single fraction.

(a) Combine the two fractions. **[4]**

(b) State the values of x excluded from the domain. **[2]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Solve the equation $\dfrac{x+2}{x-4} = 3$.

(a) Solve the equation. **[4]**

(b) Verify that your solution satisfies the necessary restriction on x. **[2]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider $f(x) = \dfrac{5x-2}{x+3}$.

(a) State the equations of the vertical and horizontal asymptotes of $y = f(x)$. **[2]**

(b) Find the coordinates of the x-intercept and the y-intercept. **[2]**

(c) Hence solve the inequality $f(x) > 0$. **[3]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Solve the inequality $\dfrac{3x+2}{x^2-2x-8} \geq 0$.

(a) Factorise the denominator. **[2]**

(b) State the critical values of x. **[2]**

(c) Hence solve the inequality. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Express $\dfrac{7x-3}{(x-2)(x+1)}$ in the form $\dfrac{A}{x-2} + \dfrac{B}{x+1}$.

(a) Set up an equation for A and B. **[3]**

(b) Solve for A and B. **[5]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Let $g(x) = \dfrac{1}{x^2-8x+15}$, where $x \in \mathbb{R}$, $x > 5$.

(a) Show that $g^{-1}(x) = 4 + \dfrac{\sqrt{x^2+x}}{x}$. **[5]**

(b) State the domain of g^{-1}. **[3]**

10. [Maximum mark: 21] *Difficulty:* 5/5

Challenge question. Consider $f(x) = \dfrac{1}{x^2-16x+63}$, where $x \in \mathbb{R}$, $x \neq 7$, $x \neq 9$.

(a) State the equations of the vertical asymptotes and the horizontal asymptote of $y = f(x)$. **[2]**

(b) For $7 < x < 9$, find the coordinates of the local maximum or minimum point of $y = f(x)$, stating which it is. **[4]**

Let $g(x) = f(x)$, where $x \in \mathbb{R}$, $x > 9$.

(c) Show that $g^{-1}(x) = 8 + \dfrac{\sqrt{x^2+x}}{x}$. **[6]**

(d) State the domain of g^{-1}. **[2]**

Let $h(x) = \arctan\left(\dfrac{x}{9}\right)$, where $x \in \mathbb{R}$.

(e) Given that $(h \circ g)(a) = \dfrac{\pi}{4}$ for some $a > 9$, find the value of a. Give your answer in the form $p + \dfrac{q}{3}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. **[7]**

Answer Key

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1. Require $x+4 \neq 0 \Rightarrow x \neq -4$.

2. (a) $x=3$. (b) $y=5$.

3. (a) $3x+6=0 \Rightarrow x=-2$; point $(-2,0)$. (b) $f(0)=\dfrac{6}{-2}=-3$; point $(0,-3)$. (c) $x=2$.

4. (a) $\dfrac{1}{x-1}+\dfrac{1}{x+4}=\dfrac{(x+4)+(x-1)}{(x-1)(x+4)}=\dfrac{2x+3}{x^2+3x-4}$. (b) $x \neq 1$ and $x \neq -4$.

5. (a) $x+2=3(x-4) \Rightarrow x+2=3x-12 \Rightarrow x=7$. (b) $x=7 \neq 4$, so the solution is valid.

6. (a) VA: $x=-3$; HA: $y=5$. (b) x-intercept: $5x-2=0 \Rightarrow x=\frac{2}{5}$, i.e. $\left(\frac{2}{5},0\right)$. y-intercept: $f(0)=-\frac{2}{3}$, i.e. $\left(0,-\frac{2}{3}\right)$. (c) Testing each interval determined by $x=-3$ and $x=\frac{2}{5}$: $f(x)>0$ for $x<-3$ or $x>\frac{2}{5}$.

7. (a) $x^2-2x-8=(x-4)(x+2)$. (b) Critical values: $x=-\frac{2}{3}$ (numerator zero), $x=4$ and $x=-2$ (denominator zero, excluded). (c) Testing each interval: $f(x) \geq 0$ for $-2<x \leq -\frac{2}{3}$ or $x>4$.

8. (a) $7x-3=A(x+1)+B(x-2)$. (b) $x=2$: $11=3A \Rightarrow A=\frac{11}{3}$. $x=-1$: $-10=-3B \Rightarrow B=\frac{10}{3}$.

9. (a) $y=\dfrac{1}{x^2-8x+15} \Rightarrow x^2-8x+15=\dfrac{1}{y} \Rightarrow x^2-8x+\left(15-\dfrac{1}{y}\right)=0$. Using the quadratic formula: $x=4\pm\sqrt{16-15+\frac{1}{y}}=4\pm\sqrt{1+\frac{1}{y}}$. Since $x>5$, take the $+$ root: $x=4+\sqrt{1+\frac{1}{y}}$. Swapping x,y: $g^{-1}(x)=4+\sqrt{1+\frac{1}{x}}=4+\dfrac{\sqrt{x^2+x}}{x}$. (b) As $x\to 5^+$, $g(x)\to+\infty$; as $x\to\infty$, $g(x)\to 0^+$. Range of g is $(0,\infty)$, so domain of g^{-1} is $x>0$.

10. (a) VA: $x=7$, $x=9$; HA: $y=0$. (b) $f'(x)=\dfrac{-(2x-16)}{(x^2-16x+63)^2}=0 \Rightarrow x=8$. At $x=8$: $f(8)=\dfrac{1}{64-128+63}=\dfrac{1}{-1}=-1$. Since $f(x)\to-\infty$ as $x\to 7^+$ or $x\to 9^-$, the point $(8,-1)$ is a local *maximum*. (c) $x=8\pm\sqrt{64-63+\frac{1}{y}}=8\pm\sqrt{1+\frac{1}{y}}$. Since $x>9$, take the $+$ root: $g^{-1}(x)=8+\sqrt{1+\frac{1}{x}}=8+\dfrac{\sqrt{x^2+x}}{x}$. (d) Range of g (for $x>9$) is $(0,\infty)$, so domain of g^{-1} is $x>0$. (e) $(h\circ g)(a)=\dfrac{\pi}{4} \Rightarrow \dfrac{g(a)}{9}=1 \Rightarrow g(a)=9 \Rightarrow \dfrac{1}{a^2-16a+63}=9 \Rightarrow a^2-16a+63=\dfrac{1}{9} \Rightarrow a^2-16a+\dfrac{566}{9}=0$. So $a=8\pm\sqrt{64-\frac{566}{9}}=8\pm\sqrt{\frac{10}{9}}=8\pm\dfrac{\sqrt{10}}{3}$. Since $a>9$, $a=8+\dfrac{1}{3}\sqrt{10}$.