Rational Functions

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

State the domain of $f(x) = \dfrac{x+2}{x-5}$. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

For $f(x) = \dfrac{3x-1}{x+2}$, state the equation of

(a) the vertical asymptote; **[2]**

(b) the horizontal asymptote. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Consider $f(x) = \dfrac{2x+4}{x-1}$.

(a) Find the coordinates of the x-intercept. **[2]**

(b) Find the coordinates of the y-intercept. **[1]**

(c) State the equation of the vertical asymptote. **[2]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Express $\dfrac{1}{x-2} + \dfrac{1}{x+3}$ as a single fraction.

(a) Combine the two fractions. **[4]**

(b) State the values of x excluded from the domain. **[2]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Solve the equation $\dfrac{x+1}{x-3} = 2$.

(a) Solve the equation. **[4]**

(b) Verify that your solution satisfies the necessary restriction on x. **[2]**

6. [Maximum mark: 7] *Difficulty: 3/5*

Consider $f(x) = \dfrac{4x-1}{x+2}$.

(a) State the equations of the vertical and horizontal asymptotes of $y = f(x)$. **[2]**

(b) Find the coordinates of the x-intercept and the y-intercept. **[2]**

(c) Hence solve the inequality $f(x) > 0$. **[3]**

7. [Maximum mark: 7] *Difficulty: 3/5*

Solve the inequality $\dfrac{2x+3}{x^2-x-6} \geq 0$.

(a) Factorise the denominator. **[2]**

(b) State the critical values of x. **[2]**

(c) Hence solve the inequality. **[3]**

8. [Maximum mark: 8] *Difficulty: 4/5*

Express $\dfrac{5x-1}{(x-1)(x+2)}$ in the form $\dfrac{A}{x-1} + \dfrac{B}{x+2}$.

(a) Set up an equation for A and B. **[3]**

(b) Solve for A and B. **[5]**

9. [Maximum mark: 8] *Difficulty: 4/5*

Let $g(x) = \dfrac{1}{x^2-4x+3}$, where $x \in \mathbb{R}$, $x > 3$.

(a) Show that $g^{-1}(x) = 2 + \dfrac{\sqrt{x^2+x}}{x}$. **[5]**

(b) State the domain of g^{-1}. **[3]**

10. [Maximum mark: 21] *Difficulty: 5/5*

Challenge question. Consider $f(x) = \dfrac{1}{x^2-6x+8}$, where $x \in \mathbb{R}$, $x \neq 2$, $x \neq 4$.

(a) State the equations of the vertical asymptotes and the horizontal asymptote of $y = f(x)$. **[2]**

(b) For $2 < x < 4$, find the coordinates of the local maximum or minimum point of $y = f(x)$, stating which it is. **[4]**

Let $g(x) = f(x)$, where $x \in \mathbb{R}$, $x > 4$.

(c) Show that $g^{-1}(x) = 3 + \dfrac{\sqrt{x^2+x}}{x}$. **[6]**

(d) State the domain of g^{-1}. **[2]**

Let $h(x) = \arctan\left(\dfrac{x}{4}\right)$, where $x \in \mathbb{R}$.

(e) Given that $(h \circ g)(a) = \dfrac{\pi}{4}$ for some $a > 4$, find the value of a. Give your answer in the form $p + \dfrac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. **[7]**

Answer Key

1. Require $x-5 \neq 0 \Rightarrow x \neq 5$.

2. (a) $x=-2$. (b) $y=3$ (ratio of leading coefficients).

3. (a) $2x+4=0 \Rightarrow x=-2$; point $(-2,0)$. (b) $f(0)=\dfrac{4}{-1}=-4$; point $(0,-4)$. (c) $x=1$.

4. (a) $\dfrac{1}{x-2}+\dfrac{1}{x+3}=\dfrac{(x+3)+(x-2)}{(x-2)(x+3)}=\dfrac{2x+1}{x^2+x-6}$. (b) $x \neq 2$ and $x \neq -3$.

5. (a) $x+1=2(x-3) \Rightarrow x+1=2x-6 \Rightarrow x=7$. (b) $x=7 \neq 3$, so the solution is valid.

6. (a) VA: $x=-2$; HA: $y=4$. (b) x-intercept: $4x-1=0 \Rightarrow x=\frac{1}{4}$, i.e. $\left(\frac{1}{4},0\right)$. y-intercept: $f(0)=-\frac{1}{2}$, i.e. $\left(0,-\frac{1}{2}\right)$. (c) Testing each interval determined by $x=-2$ and $x=\frac{1}{4}$: $f(x)>0$ for $x<-2$ or $x>\frac{1}{4}$.

7. (a) $x^2-x-6=(x-3)(x+2)$. (b) Critical values: $x=-\frac{3}{2}$ (numerator zero), $x=3$ and $x=-2$ (denominator zero, excluded). (c) Testing each interval: $f(x) \geq 0$ for $-2<x \leq -\frac{3}{2}$ or $x>3$.

8. (a) $5x-1=A(x+2)+B(x-1)$. (b) $x=1$: $4=3A \Rightarrow A=\frac{4}{3}$. $x=-2$: $-11=-3B \Rightarrow B=\frac{11}{3}$.

9. (a) $y=\dfrac{1}{x^2-4x+3} \Rightarrow x^2-4x+3=\dfrac{1}{y} \Rightarrow x^2-4x+\left(3-\dfrac{1}{y}\right)=0$. Using the quadratic formula: $x=2 \pm \sqrt{4-3+\frac{1}{y}}=2 \pm \sqrt{1+\frac{1}{y}}$. Since $x>3$, we require the $+$ root, so $x=2+\sqrt{1+\frac{1}{y}}$. Swapping x,y: $g^{-1}(x)=2+\sqrt{1+\frac{1}{x}}=2+\dfrac{\sqrt{x^2+x}}{x}$. (b) Domain of g^{-1} = range of g. As $x \to 3^+$, $g(x) \to +\infty$; as $x \to \infty$, $g(x) \to 0^+$. So the range of g is $(0,\infty)$, giving domain of g^{-1}: $x>0$.

10. (a) VA: $x=2, x=4$; HA: $y=0$. (b) $f'(x)=\dfrac{-(2x-6)}{(x^2-6x+8)^2}=0 \Rightarrow x=3$. At $x=3$: $f(3)=\dfrac{1}{9-18+8}=\dfrac{1}{-1}=-1$. Since $f(x) \to -\infty$ as $x \to 2^+$ or $x \to 4^-$, the point $(3,-1)$ is a local *maximum*. (c) Following the same method as Question 9 with x^2-6x+8: $x=3 \pm \sqrt{9-8+\frac{1}{y}}=3 \pm \sqrt{1+\frac{1}{y}}$. Since $x>4$, take the $+$ root: $g^{-1}(x)=3+\sqrt{1+\frac{1}{x}}=3+\dfrac{\sqrt{x^2+x}}{x}$. (d) Range of g (for $x>4$) is $(0,\infty)$, so domain of g^{-1} is $x>0$. (e) $(h \circ g)(a)=\dfrac{\pi}{4} \Rightarrow \dfrac{g(a)}{4}=\tan\dfrac{\pi}{4}=1 \Rightarrow g(a)=4 \Rightarrow \dfrac{1}{a^2-6a+8}=4 \Rightarrow a^2-6a+8=\dfrac{1}{4} \Rightarrow a^2-6a+\dfrac{31}{4}=0$. So $a=3 \pm \sqrt{9-\frac{31}{4}}=3 \pm \sqrt{\frac{5}{4}}=3 \pm \dfrac{\sqrt{5}}{2}$. Since $a>4$, $a=3+\dfrac{1}{2}\sqrt{5}$.