

Probability

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

1. [Maximum mark: 4]

Difficulty: 1/5

A discrete random variable X has probability distribution given by the following table.

x	1	2	3	4
$P(X = x)$	k	0.3	0.25	0.2

- (a) Find the value of k . [2]
- (b) Hence find $P(X \geq 2)$. [2]

2. [Maximum mark: 5]

Difficulty: 1/5

A discrete random variable X has probability distribution given by $P(X = 1) = 3k$, $P(X = 2) = k$, $P(X = 3) = k$.

- (a) Find the value of k . [2]
- (b) Find $E(X)$. [3]

3. [Maximum mark: 6]

Difficulty: 2/5

Events A and B are such that $P(A) = 0.7$ and $P(B) = 0.5$.

- (a) Determine the value of $P(A \cap B)$ in the case where A and B are independent. [2]
- (b) Determine the minimum possible value of $P(A \cap B)$. [2]
- (c) Determine the maximum possible value of $P(A \cap B)$. [2]

4. [Maximum mark: 6]

Difficulty: 2/5

Events A and B are independent and $P(A) = 5P(B)$.

Given that $P(A \cup B) = 0.55$, find $P(B)$. [6]

5. [Maximum mark: 6]

Difficulty: 2/5

Let A and B be two independent events such that $P(A \cap B') = 0.01$ and $P(A' \cap B) = 0.81$.

(a) Given that $P(A \cap B) = x$, find the value of x . [4]

(b) Find $P(A'|B')$. [2]

6. [Maximum mark: 7]

Difficulty: 3/5

A discrete random variable X has the following probability distribution.

x	0	1	2	3
$P(X = x)$	0.20	$k - 0.06$	0.45	$0.25 - k^2$

(a) Show that $k^2 - k + 0.16 = 0$. [1]

(b) Find the value of k , giving a reason for your answer. [3]

(c) Hence find $E(X)$. [3]

7. [Maximum mark: 8]

Difficulty: 4/5

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} kx(b-x), & 0 \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

where $k, b \in \mathbb{R}^+$.

(a) Find an expression for k in terms of b . [5]

(b) In the case where $b = 6$, find the median of X . [3]

8. [Maximum mark: 16]

Difficulty: 4/5

A biased four-sided die, A , is rolled. Let X be the score obtained when die A is rolled. The probability distribution for X is given in the following table.

x	1	2	3	4
$P(X = x)$	$2p$	p	p	p

(a) Find the value of p . [2]

(b) Hence find $E(X)$. [2]

A second biased four-sided die, B , is rolled. Let Y be the score obtained when die B is rolled. The probability distribution for Y is given in the following table.

y	1	2	3	4
$P(Y = y)$	q	q	$2q$	r

(c) State the range of possible values of q . [3]

(d) Hence find the range of possible values of r . [3]

Marco rolls die A once and Nadia rolls die B once. The probability that Marco's score is less than Nadia's score is $\frac{16}{25}$.

(e) Find the value of $E(Y)$. [6]

9. [Maximum mark: 7]

Difficulty: 4/5

Yuki plays a game with die A from Question 8. In this game she is allowed a maximum of four rolls. Her score is calculated by adding the results of each roll. Yuki wins the game if her score is at least 7.

After two rolls of the die, Yuki has a score of 2.

- (a) Assuming that rolls of the die are independent, find the probability that Yuki wins the game. [7]

10. [Maximum mark: 15]

Difficulty: 5/5

Challenge question. Leila has two bags of marbles. Bag 1 contains 4 red and 4 blue marbles. Bag 2 contains 1 red and 5 blue marbles. Leila selects Bag 1 with probability $\frac{3}{5}$ and Bag 2 with probability $\frac{2}{5}$, and then draws one marble from the selected bag.

- (a) Find the probability that the marble drawn is red. [4]

- (b) Given that the marble drawn is red, find the probability that it came from Bag 1. [5]

The marble drawn is *not* replaced. A second marble is then drawn from the same bag.

- (c) Given that the first marble drawn was red, find the probability that the second marble drawn is also red. [6]

Answer Key

1. (a) $k + 0.3 + 0.25 + 0.2 = 1 \Rightarrow k = 0.25$. (b) $P(X \geq 2) = 0.3 + 0.25 + 0.2 = 0.75$.

2. (a) $3k + k + k = 1 \Rightarrow 5k = 1 \Rightarrow k = 0.2$. (b) $E(X) = 1(0.6) + 2(0.2) + 3(0.2) = 0.6 + 0.4 + 0.6 = 1.6$.

3. (a) $P(A \cap B) = P(A)P(B) = 0.7(0.5) = 0.35$. (b) Minimum = $\max(0, 0.7 + 0.5 - 1) = \max(0, 0.2) = 0.2$.
(c) Maximum = $\min(0.7, 0.5) = 0.5$.

4. Let $P(B) = x$, so $P(A) = 5x$ and $P(A \cap B) = 5x^2$. Then $P(A \cup B) = 5x + x - 5x^2 = 6x - 5x^2 = 0.55 \Rightarrow 5x^2 - 6x + 0.55 = 0$. Solving: $x = \frac{6 \pm \sqrt{36 - 11}}{10} = \frac{6 \pm 5}{10}$, giving $x = 1.1$ (rejected, $P(A) > 1$) or $x = 0.1$. So $P(B) = 0.1$.

5. (a) By independence, $P(A)(1 - P(B)) = 0.01$ and $(1 - P(A))P(B) = 0.81$. Subtracting: $P(A) - P(B) = 0.01 - 0.81 = -0.8$, so $P(B) = P(A) + 0.8$. Substituting: $P(A)(1 - P(A) - 0.8) = 0.01 \Rightarrow P(A)(0.2 - P(A)) = 0.01 \Rightarrow P(A)^2 - 0.2P(A) + 0.01 = 0 \Rightarrow (P(A) - 0.1)^2 = 0 \Rightarrow P(A) = 0.1$, so $P(B) = 0.9$. Hence $x = 0.1(0.9) = 0.09$. (b) $P(A' \cap B') = 1 - P(A \cup B) = 1 - (0.1 + 0.9 - 0.09) = 0.09$. $P(B') = 1 - 0.9 = 0.1$. So $P(A'|B') = \frac{0.09}{0.1} = 0.9$.

6. (a) Summing to 1: $0.20 + (k - 0.06) + 0.45 + (0.25 - k^2) = 1 \Rightarrow 0.84 + k - k^2 = 1 \Rightarrow k^2 - k + 0.16 = 0$.
(b) Solving: $k = \frac{1 \pm \sqrt{1 - 0.64}}{2} = \frac{1 \pm 0.6}{2}$, giving $k = 0.8$ or $k = 0.2$. At $k = 0.8$: $P(X = 3) = 0.25 - 0.64 = -0.39 < 0$, invalid. So $k = 0.2$. (c) Probabilities: 0.20, 0.14, 0.45, 0.21. $E(X) = 0(0.20) + 1(0.14) + 2(0.45) + 3(0.21) = 0 + 0.14 + 0.90 + 0.63 = 1.67$.

7. (a) $\int_0^b kx(b-x)dx = k \left[\frac{bx^2}{2} - \frac{x^3}{3} \right]_0^b = k \left(\frac{b^3}{2} - \frac{b^3}{3} \right) = \frac{kb^3}{6} = 1 \Rightarrow k = \frac{6}{b^3}$. (b) Since f is symmetric about $x = \frac{b}{2}$ (as $f(x) = f(b-x)$), the median is $m = \frac{b}{2}$. With $b = 6$: $m = 3$.

8. (a) $2p + p + p + p = 1 \Rightarrow 5p = 1 \Rightarrow p = 0.2$. (b) $E(X) = 1(0.4) + 2(0.2) + 3(0.2) + 4(0.2) = 0.4 + 0.4 + 0.6 + 0.8 = 2.2$. (c) Since $4q + r = 1$ with $q, r > 0$: $r = 1 - 4q > 0 \Rightarrow q < \frac{1}{4}$. So $0 < q < \frac{1}{4}$. (d) As q ranges over $(0, \frac{1}{4})$, $r = 1 - 4q$ ranges over $(0, 1)$. (e) Computing $P(X < Y)$ in terms of q (using $p = 0.2$ and $r = 1 - 4q$) gives $P(X < Y) = \frac{4}{5} - \frac{8q}{5}$. Setting this equal to $\frac{16}{25}$: $\frac{4}{5} - \frac{8q}{5} = \frac{16}{25} \Rightarrow \frac{8q}{5} = \frac{4}{25} \Rightarrow \frac{8q}{5} = \frac{4}{25} \Rightarrow \frac{20}{25} - \frac{16}{25} = \frac{4}{25} \Rightarrow q = \frac{1}{10}$. Then $r = 1 - 4(\frac{1}{10}) = \frac{3}{5}$, and $E(Y) = 1(q) + 2(q) + 3(2q) + 4(r) = 9q + 4r = 9(\frac{1}{10}) + 4(\frac{3}{5}) = \frac{9}{10} + \frac{24}{10} = \frac{33}{10}$.

9. Yuki needs the sum of her remaining two rolls to be at least $7 - 2 = 5$. Using the distribution of die A ($P(X=1, 2, 3, 4) = 0.4, 0.2, 0.2, 0.2$), the distribution of the sum of two independent rolls gives $P(\text{sum} \geq 5) = \frac{12}{25}$.

10. (a) $P(\text{red}) = \frac{3}{5} \left(\frac{4}{8} \right) + \frac{2}{5} \left(\frac{1}{6} \right) = \frac{3}{10} + \frac{1}{15} = \frac{9}{30} + \frac{2}{30} = \frac{11}{30}$. (b) $P(\text{Bag 1} | \text{red}) = \frac{\frac{3}{10}}{\frac{11}{30}} = \frac{3}{10} \times \frac{30}{11} = \frac{9}{11}$. (c) $P(\text{Bag 2} | \text{red}) = 1 - \frac{9}{11} = \frac{2}{11}$. Given the bag, the second draw is red with probability $\frac{3}{7}$ (Bag 1: 3 red left out of 7) or 0 (Bag 2: 0 red left out of 5, since its only red marble was already drawn). So $P(\text{2nd red} | \text{1st red}) = \frac{9}{11} \cdot \frac{3}{7} + \frac{2}{11} \cdot 0 = \frac{27}{77}$.