Probability

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

A discrete random variable X has probability distribution given by the following table.

x	1	2	3	4
$P(X=x)$	0.25	k	0.2	0.3

(a) Find the value of k. **[2]**

(b) Hence find $P(X<3)$. **[2]**

2. [Maximum mark: 5] *Difficulty:* 1/5

A discrete random variable X has probability distribution given by $P(X=1)=k$, $P(X=2)=2k$, $P(X=3)=2k$.

(a) Find the value of k. **[2]**

(b) Find $E(X)$. **[3]**

3. [Maximum mark: 6] *Difficulty:* 2/5

Events A and B are such that $P(A)=0.3$ and $P(B)=0.5$.

(a) Determine the value of $P(A\cap B)$ in the case where A and B are independent. **[2]**

(b) Determine the minimum possible value of $P(A\cap B)$. **[2]**

(c) Determine the maximum possible value of $P(A\cap B)$. **[2]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Events A and B are independent and $P(A)=4P(B)$.

Given that $P(A\cup B)=0.84$, find $P(B)$. **[6]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Let A and B be two independent events such that $P(A\cap B')=0.04$ and $P(A'\cap B)=0.64$.

(a) Given that $P(A \cap B) = x$, find the value of x. **[4]**

(b) Find $P(A'|B')$. **[2]**

6. [Maximum mark: 7] *Difficulty: 3/5*

A discrete random variable X has the following probability distribution.

x	0	1	2	3
$P(X = x)$	0.25	$k - 0.03$	0.40	$0.30 - 2k^2$

(a) Show that $2k^2 - k + 0.08 = 0$. **[1]**

(b) Find the value of k, giving a reason for your answer. **[3]**

(c) Hence find $E(X)$. **[3]**

7. [Maximum mark: 8] *Difficulty: 4/5*

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} k(b-x), & 0 \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

where $k, b \in \mathbb{R}^+$.

(a) Find an expression for k in terms of b. **[5]**

(b) In the case where $b = 4$, find the median of X. **[3]**

8. [Maximum mark: 16] *Difficulty: 4/5*

A biased four-sided die, A, is rolled. Let X be the score obtained when die A is rolled. The probability distribution for X is given in the following table.

x	1	2	3	4
$P(X = x)$	p	p	$3p$	p

(a) Find the value of p. **[2]**

(b) Hence find $E(X)$. **[2]**

A second biased four-sided die, B, is rolled. Let Y be the score obtained when die B is rolled. The probability distribution for Y is given in the following table.

y	1	2	3	4
$P(Y = y)$	$2q$	q	q	r

(c) State the range of possible values of q. **[3]**

(d) Hence find the range of possible values of r. **[3]**

Priya rolls die A once and Quinn rolls die B once. The probability that Priya's score is less than Quinn's score is $\frac{13}{36}$.

(e) Find the value of $E(Y)$. **[6]**

9. [Maximum mark: 7] *Difficulty: 4/5*

Diego plays a game with die A from Question 8. In this game he is allowed a maximum of four rolls. His score is calculated by adding the results of each roll. Diego wins the game if his score is at least 9.

After two rolls of the die, Diego has a score of 4.

(a) Assuming that rolls of the die are independent, find the probability that Diego wins the game. **[7]**

10. [Maximum mark: 15] *Difficulty: 5/5*

Challenge question. Farah has two bags of marbles. Bag 1 contains 2 red and 6 blue marbles. Bag 2 contains 5 red and 3 blue marbles. Farah selects Bag 1 with probability $\frac{2}{5}$ and Bag 2 with probability $\frac{3}{5}$, and then draws one marble from the selected bag.

(a) Find the probability that the marble drawn is red. **[4]**

(b) Given that the marble drawn is red, find the probability that it came from Bag 1. **[5]**

The marble drawn is *not* replaced. A second marble is then drawn from the same bag.

(c) Given that the first marble drawn was red, find the probability that the second marble drawn is also red. **[6]**

Answer Key

1. (a) $0.25+k+0.2+0.3=1 \Rightarrow k=0.25$. (b) $P(X<3)=0.25+0.25=0.5$.

2. (a) $k+2k+2k=1 \Rightarrow 5k=1 \Rightarrow k=0.2$. (b) $E(X)=1(0.2)+2(0.4)+3(0.4)=0.2+0.8+1.2=2.2$.

3. (a) $P(A\cap B)=P(A)P(B)=0.3(0.5)=0.15$. (b) Minimum $=\max(0,0.3+0.5-1)=\max(0,-0.2)=0$. (c) Maximum $=\min(0.3,0.5)=0.3$.

4. Let $P(B)=x$, so $P(A)=4x$ and $P(A\cap B)=4x^2$. Then $P(A\cup B)=4x+x-4x^2=5x-4x^2=0.84 \Rightarrow 4x^2-5x+0.84=0$. Solving: $x=\dfrac{5\pm\sqrt{25-13.44}}{8}=\dfrac{5\pm 3.4}{8}$, giving $x=1.05$ (rejected, $P(A)>1$) or $x=0.2$. So $P(B)=0.2$.

5. (a) By independence, $P(A)(1-P(B))=0.04$ and $(1-P(A))P(B)=0.64$. Subtracting: $P(A)-P(B)=0.04-0.64=-0.6$, so $P(B)=P(A)+0.6$. Substituting: $P(A)(1-P(A)-0.6)=0.04 \Rightarrow P(A)(0.4-P(A))=0.04 \Rightarrow P(A)^2-0.4P(A)+0.04=0 \Rightarrow (P(A)-0.2)^2=0 \Rightarrow P(A)=0.2$, so $P(B)=0.8$. Hence $x=0.2(0.8)=0.16$. (b) $P(A'\cap B')=1-P(A\cup B)=1-(0.2+0.8-0.16)=0.16$. $P(B')=1-0.8=0.2$. So $P(A'|B')=\dfrac{0.16}{0.2}=0.8$.

6. (a) Summing to 1: $0.25+(k-0.03)+0.40+(0.30-2k^2)=1 \Rightarrow 0.92+k-2k^2=1 \Rightarrow 2k^2-k+0.08=0$. (b) Solving: $k=\dfrac{1\pm\sqrt{1-0.64}}{4}=\dfrac{1\pm 0.6}{4}$, giving $k=0.4$ or $k=0.1$. At $k=0.4$: $P(X=3)=0.30-2(0.16)=-0.02<0$, invalid. So $k=0.1$. (c) Probabilities: $0.25, 0.07, 0.40, 0.28$. $E(X)=0(0.25)+1(0.07)+2(0.40)+3(0.28)=0+0.07+0.80+0.84=1.71$.

7. (a) $\displaystyle\int_0^b k(b-x)\,dx=k\left[bx-\frac{x^2}{2}\right]_0^b=k\frac{b^2}{2}=1 \Rightarrow k=\frac{2}{b^2}$. (b) With $b=4$: $k=\dfrac{1}{8}$. Median m satisfies $\dfrac{1}{8}\left[4m-\dfrac{m^2}{2}\right]=0.5 \Rightarrow 4m-\dfrac{m^2}{2}=4 \Rightarrow m^2-8m+8=0 \Rightarrow m=\dfrac{8\pm\sqrt{32}}{2}=4\pm 2\sqrt{2}$. Since $0\le m\le 4$, $m=4-2\sqrt{2}\approx 1.17$.

8. (a) $p+p+3p+p=1 \Rightarrow 6p=1 \Rightarrow p=\dfrac{1}{6}$. (b) $E(X)=1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{2}\right)+4\left(\frac{1}{6}\right)=\dfrac{1}{6}+\dfrac{2}{6}+\dfrac{9}{6}+\dfrac{4}{6}=\dfrac{16}{6}=\dfrac{8}{3}$. (c) Since $4q+r=1$ with $q,r>0$: $r=1-4q>0 \Rightarrow q<\dfrac{1}{4}$. So $0<q<\dfrac{1}{4}$. (d) As q ranges over $\left(0,\frac{1}{4}\right)$, $r=1-4q$ ranges over $(0,1)$. (e) Computing $P(X<Y)$ in terms of q (using $p=\frac{1}{6}$ and $r=1-4q$) gives $P(X<Y)=\dfrac{5}{6}-\dfrac{17q}{6}$. Setting this equal to $\dfrac{13}{36}$: $\dfrac{5}{6}-\dfrac{17q}{6}=\dfrac{13}{36} \Rightarrow \dfrac{17q}{6}=\dfrac{30}{36}-\dfrac{13}{36}=\dfrac{17}{36} \Rightarrow q=\dfrac{1}{6}$. Then $r=1-4\left(\frac{1}{6}\right)=\dfrac{1}{3}$, and $E(Y)=1(2q)+2(q)+3(q)+4(r)=7q+4r=7\left(\frac{1}{6}\right)+4\left(\frac{1}{3}\right)=\dfrac{7}{6}+\dfrac{4}{3}=\dfrac{7}{6}+\dfrac{8}{6}=\dfrac{15}{6}=\dfrac{5}{2}$.

9. Diego needs the sum of his remaining two rolls to be at least $9-4=5$. Using the distribution of die A ($P(X{=}1,2,3,4)=\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{6}$), the distribution of the sum of two independent rolls gives $P(\text{sum}\ge 5)=\dfrac{13}{18}$.

10. (a) $P(\text{red})=\dfrac{2}{5}\left(\dfrac{2}{8}\right)+\dfrac{3}{5}\left(\dfrac{5}{8}\right)=\dfrac{1}{10}+\dfrac{3}{8}=\dfrac{4}{40}+\dfrac{15}{40}=\dfrac{19}{40}$. (b) $P(\text{Bag 1}|\text{red})=\dfrac{\frac{1}{10}}{\frac{19}{40}}=\dfrac{1}{10}\times\dfrac{40}{19}=\dfrac{4}{19}$. (c) $P(\text{Bag 2}|\text{red})=1-\dfrac{4}{19}=\dfrac{15}{19}$. Given the bag, the second draw is red with probability $\dfrac{1}{7}$ (Bag 1: 1 red left out of 7) or $\dfrac{4}{7}$ (Bag 2: 4 red left out of 7). So $P(\text{2nd red}|\text{1st red})=\dfrac{4}{19}\cdot\dfrac{1}{7}+\dfrac{15}{19}\cdot\dfrac{4}{7}=\dfrac{4}{133}+\dfrac{60}{133}=\dfrac{64}{133}$.