The Normal Distribution

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

The random variable $X \sim N(120, 15^2)$.

(a) Find $P(X < 100)$. **[2]**

(b) Find $P(X > 135)$. **[2]**

2. [Maximum mark: 5] *Difficulty:* 1/5

The random variable $X \sim N(500, 60^2)$. Find the value of a such that $P(X > a) = 0.05$. **[5]**

3. [Maximum mark: 5] *Difficulty:* 2/5

The random variable $X \sim N(\mu, 12^2)$. Given that $P(X < 90) = 0.10$, find the value of μ. **[5]**

4. [Maximum mark: 6] *Difficulty:* 2/5

The random variable $X \sim N(200, 25^2)$. Find $P(180 < X < 230)$. **[6]**

5. [Maximum mark: 7] *Difficulty:* 3/5

The masses of packets of flour produced in a factory are normally distributed with mean 500 g and standard deviation 15 g. A packet is graded "underweight" if its mass is less than 480 g.

(a) Find the probability that a randomly selected packet is graded underweight. **[2]**

(b) In a random sample of 10 packets, find the probability that exactly 4 are graded underweight. **[5]**

6. [Maximum mark: 8] *Difficulty:* 3/5

A dairy produces two types of cheese. Type A blocks have mass M_A grams, where $M_A \sim N(30, 2^2)$. Type B blocks have mass M_B grams, where $M_B \sim N(35, 3^2)$. Of all blocks produced, 55% are Type A and 45% are Type B.

A block is selected at random from all those produced.

(a) Find the probability that its mass is less than 28 g. **[4]**

(b) Given that its mass is less than 28 g, find the probability that it is Type A. **[4]**

7. [Maximum mark: 7] *Difficulty:* 4/5

The machine producing Type A blocks (from Question 6) is adjusted so that the mean mass remains 30 g, but the standard deviation changes to σ g. The machine producing Type B blocks is not adjusted. The probability that the mass of a randomly selected block (from these adjusted machines, in the same 55%/45% proportions) is less than 28 g is now 0.0546.

Find the value of σ. **[7]**

8. [Maximum mark: 8] *Difficulty:* 4/5

The masses of two independent items are given by $X \sim N(12, 2.5^2)$ kg and $Y \sim N(9, 1.5^2)$ kg. Find $P(X+Y < 18)$. **[8]**

9. [Maximum mark: 9] *Difficulty:* 4/5

The random variable $X \sim N(\mu, \sigma^2)$ is such that $P(X < 70) = 0.15$ and $P(X < 100) = 0.85$.

Find the value of μ and the value of σ. **[9]**

10. [Maximum mark: 16] *Difficulty:* 5/5

Challenge question. Two high-jump athletes, Omar and Priya, are competing. The heights, O metres, jumped by Omar can be modelled by a normal distribution with mean 1.55 and standard deviation 0.06. The heights, P metres, jumped by Priya can be modelled by a normal distribution with mean 1.60 and standard deviation 0.05.

In the first round of competition, each athlete has six jumps. To qualify for the next round, an athlete must record at least one jump of 1.61 metres or greater in the first round.

(a) Find the probability that Omar qualifies for the next round. **[4]**

(b) Find the probability that Priya qualifies for the next round. **[3]**

(c) Find the probability that exactly one of Omar or Priya qualifies for the next round. **[4]**

Suppose instead that Omar is allowed n jumps in the first round (still needing at least one jump of 1.61 metres or greater to qualify).

(d) Find the least value of n for which Omar's probability of qualifying is at least 0.9. **[5]**

Answer Key

1. (a) $P(X < 100) = P\left(Z < \frac{100-120}{15}\right) = P(Z < -1.333) \approx 0.0912$. (b) $P(X > 135) = P\left(Z > \frac{135-120}{15}\right) = P(Z > 1) \approx 0.1587$.

2. $P(X > a) = 0.05 \Rightarrow P(X < a) = 0.95 \Rightarrow \frac{a-500}{60} = z_{0.95} \approx 1.6449 \Rightarrow a \approx 598.7$.

3. $P(X < 90) = 0.10 \Rightarrow \frac{90-\mu}{12} = z_{0.10} \approx -1.2816 \Rightarrow \mu \approx 105.4$.

4. $P(180 < X < 230) = P(X < 230) - P(X < 180) \approx 0.8849 - 0.2119 \approx 0.673$.

5. (a) $P(\text{mass} < 480) = P\left(Z < \frac{480-500}{15}\right) = P(Z < -1.333) \approx 0.0912$. (b) Using $B \sim \text{Bin}(10, 0.0912)$: $P(B = 4) = \binom{10}{4}(0.0912)^4(0.9088)^6 \approx 0.00819$.

6. (a) $P(M_A < 28) = P\left(Z < \frac{28-30}{2}\right) \approx 0.1587$. $P(M_B < 28) = P\left(Z < \frac{28-35}{3}\right) \approx 0.00982$. $P(\text{mass} < 28) = 0.55(0.1587) + 0.45(0.00982) \approx 0.0917$. (b) $P(\text{Type } A \mid \text{mass} < 28) = \frac{0.55(0.1587)}{0.0917} \approx 0.952$.

7. Let $p = P(M_A < 28)$ with the new σ. Then $0.55p + 0.45(0.00982) = 0.0546 \Rightarrow 0.55p = 0.0546 - 0.00442 = 0.0502 \Rightarrow p \approx 0.0913$. So $\frac{28-30}{\sigma} = z_{0.0913} \approx -1.333 \Rightarrow \sigma \approx 1.50$.

8. Since $X + Y \sim N(21, 2.5^2 + 1.5^2) = N(21, 8.5)$: $P(X + Y < 18) = P\left(Z < \frac{18-21}{\sqrt{8.5}}\right) = P(Z < -1.029) \approx 0.152$.

9. $\frac{70-\mu}{\sigma} = z_{0.15} \approx -1.0364$ and $\frac{100-\mu}{\sigma} = z_{0.85} \approx 1.0364$. Subtracting: $\frac{30}{\sigma} = 2.0728 \Rightarrow \sigma \approx 14.5$. By symmetry of the two equations, $\mu = 85.0$.

10. (a) $P(\text{single jump} \geq 1.61) = P\left(Z \geq \frac{1.61-1.55}{0.06}\right) = P(Z \geq 1) \approx 0.1587$. $P(\text{Omar qualifies}) = 1 - (1-0.1587)^6 \approx 0.646$. (b) $P(\text{single jump} \geq 1.61) = P\left(Z \geq \frac{1.61-1.60}{0.05}\right) = P(Z \geq 0.2) \approx 0.4207$. $P(\text{Priya qualifies}) = 1 - (1-0.4207)^6 \approx 0.962$. (c) $P(\text{exactly one qualifies}) = 0.646(1-0.962) + (1-0.646)(0.962) \approx 0.0245 + 0.3405 \approx 0.365$. (d) Need $1 - (1-0.1587)^n \geq 0.9 \Rightarrow (0.8413)^n \leq 0.1 \Rightarrow n \geq \frac{\ln 0.1}{\ln 0.8413} \approx 13.33$. So the least integer value is $n = 14$ (check: $n = 13$ gives $P \approx 0.895 < 0.9$; $n = 14$ gives $P \approx 0.912 \geq 0.9$).