Mathematical Proof

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Prove directly that the sum of an even integer and an odd integer is odd. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Prove directly that if n is an odd integer, then n^2 is odd. **[4]**

3. [Maximum mark: 4] *Difficulty:* 2/5

Disprove the following statement by giving a counterexample:

"For all integers $n \geq 1$, $n^2 - n + 41$ is prime."

[4]

4. [Maximum mark: 5] *Difficulty:* 2/5

Prove by contrapositive that, for $n \in \mathbb{Z}$, if n^2 is divisible by 5 then n is divisible by 5. **[5]**

5. [Maximum mark: 5] *Difficulty:* 2/5

Prove directly that $x^2 - 8x + 20 > 0$ for all real values of x. **[5]**

6. [Maximum mark: 6] *Difficulty:* 3/5

Prove by contradiction that there is no smallest positive real number. **[6]**

7. [Maximum mark: 6] *Difficulty:* 3/5

Prove by contradiction that $\sqrt{7}$ is irrational. **[6]**

8. [Maximum mark: 7] *Difficulty:* 4/5

Consider integers p and q such that $p^2 - q^2$ is exactly divisible by 4. Prove by contradiction that p and q must have the same parity (both even, or both odd). **[7]**

9. [Maximum mark: 7] *Difficulty:* 4/5

Prove by contradiction that the equation $4x^3 + 8x + 1 = 0$ has no integer roots. **[7]**

10. [Maximum mark: 9] *Difficulty:* 5/5

Challenge question. Prove by contradiction that $\log_2 5$ is irrational.

(a) Assume $\log_2 5$ is rational and equal to $\dfrac{p}{q}$ for some positive integers p, q. Show that this leads to $2^p = 5^q$. **[4]**

(b) By considering the parity of 2^p and 5^q, complete the proof. **[5]**

Answer Key

1. Let $a = 2m$ (even) and $b = 2n+1$ (odd) for some $m,n \in \mathbb{Z}$. Then $a+b = 2m+2n+1 = 2(m+n)+1$, which is odd. Hence the sum of an even integer and an odd integer is odd. ■

2. Let $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $n^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$, which is odd. ■

3. Let $n = 41$: $n^2 - n + 41 = 41^2 - 41 + 41 = 41^2 = 41 \times 41$, which is not prime (it has a factor of 41 other than itself and 1, in the sense that it is a perfect square of a prime, so is composite). This single counterexample disproves the statement.

4. The contrapositive is: if n is not divisible by 5, then n^2 is not divisible by 5. If n is not divisible by 5, then $n \equiv 1, 2, 3 \text{ or } 4 \pmod 5$. Squaring: $1^2 \equiv 1$, $2^2 \equiv 4$, $3^2 \equiv 4$, $4^2 \equiv 1 \pmod 5$. In every case $n^2 \not\equiv 0 \pmod 5$, so n^2 is not divisible by 5. Since the contrapositive is true, the original statement is true. ■

5. Completing the square: $x^2 - 8x + 20 = (x-4)^2 + 4$. Since $(x-4)^2 \geq 0$ for all real x, we have $(x-4)^2 + 4 \geq 4 > 0$. Hence $x^2 - 8x + 20 > 0$ for all real x. ■

6. Suppose, for contradiction, that a smallest positive real number r exists. Then $\dfrac{r}{2}$ is also a real number, and since $r > 0$, we have $0 < \dfrac{r}{2} < r$. So $\dfrac{r}{2}$ is a positive real number smaller than r, contradicting the assumption that r is smallest. Hence no smallest positive real number exists. ■

7. Suppose, for contradiction, that $\sqrt{7}$ is rational, so $\sqrt{7} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ share no common factor and $q \neq 0$. Then $p^2 = 7q^2$, so $7 \mid p^2$, and since 7 is prime, $7 \mid p$. Write $p = 7k$. Then $49k^2 = 7q^2 \Rightarrow 7k^2 = q^2$, so $7 \mid q^2$, and hence $7 \mid q$. But then 7 divides both p and q, contradicting that they share no common factor. Hence $\sqrt{7}$ is irrational. ■

8. Suppose, for contradiction, that p and q have opposite parity; without loss of generality let $p = 2a$ (even) and $q = 2b+1$ (odd) for some $a, b \in \mathbb{Z}$. Then

$$p^2 - q^2 = 4a^2 - (2b+1)^2 = 4a^2 - 4b^2 - 4b - 1 = 4(a^2 - b^2 - b) - 1,$$

which leaves remainder 3 on division by 4 (since $-1 \equiv 3 \pmod 4$), contradicting the assumption that $p^2 - q^2$ is exactly divisible by 4. Hence p and q must have the same parity. ■

9. Suppose, for contradiction, that $x = k$ is an integer root of $4x^3 + 8x + 1 = 0$. Then $4k^3 + 8k = -1$, i.e. $4(k^3 + 2k) = -1$. The left-hand side is divisible by 4 (since $k^3 + 2k \in \mathbb{Z}$), but -1 is not divisible by 4. This is a contradiction. Hence the equation has no integer roots. ■

10. (a) Since $\log_2 5 > 0$, we may assume $\log_2 5 = \dfrac{p}{q}$ for some positive integers p, q. Then $2^{p/q} = 5$, and raising both sides to the power q gives $2^p = 5^q$. (b) Since $p \geq 1$, 2^p is even. Since $q \geq 1$, 5^q is a product of odd numbers and so is odd. But $2^p = 5^q$ would then equate an even number to an odd number, which is impossible. This contradiction shows that $\log_2 5$ cannot be rational; hence $\log_2 5$ is irrational. ■