Mathematical Proof

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Prove directly that the sum of any two odd integers is even. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Prove directly that if n is an even integer, then n^2 is divisible by 4. **[4]**

3. [Maximum mark: 4] *Difficulty:* 2/5

Disprove the following statement by giving a counterexample:

"For all integers $n \geq 1,\ 2^n + 1$ is prime."

[4]

4. [Maximum mark: 5] *Difficulty:* 2/5

Prove by contrapositive that, for $n \in \mathbb{Z}$, if n^2 is divisible by 3 then n is divisible by 3. **[5]**

5. [Maximum mark: 5] *Difficulty:* 2/5

Prove directly that $x^2 + 6x + 10 \geq 1$ for all real values of x. **[5]**

6. [Maximum mark: 6] *Difficulty:* 3/5

Prove by contradiction that there is no largest negative rational number. **[6]**

7. [Maximum mark: 6] *Difficulty:* 3/5

Prove by contradiction that $\sqrt{5}$ is irrational. **[6]**

8. [Maximum mark: 7] *Difficulty:* 4/5

Consider integers p and q such that $p^2 + q^2$ is exactly divisible by 4. Prove by contradiction that p and q must both be even. **[7]**

9. [Maximum mark: 7] *Difficulty:* 4/5

Prove by contradiction that the equation $5x^3 + 10x + 3 = 0$ has no integer roots. **[7]**

10. [Maximum mark: 9] *Difficulty:* 5/5

Challenge question. Prove by contradiction that $\log_3 2$ is irrational.

(a) Assume $\log_3 2$ is rational and equal to $\frac{p}{q}$ for some positive integers p, q. Show that this leads to $3^p = 2^q$. **[4]**

(b) By considering the parity of 3^p and 2^q, complete the proof. **[5]**

Answer Key

1. Let $a = 2m+1$ and $b = 2n+1$ for some $m,n \in \mathbb{Z}$. Then $a+b = 2m+2n+2 = 2(m+n+1)$, which is even since $m+n+1 \in \mathbb{Z}$. Hence the sum of two odd integers is even. ∎

2. Let $n = 2m$ for some $m \in \mathbb{Z}$. Then $n^2 = 4m^2$, which is clearly divisible by 4. ∎

3. Let $n = 5$: $2^5+1 = 33 = 3\times 11$, which is not prime. This single counterexample disproves the statement.

4. The contrapositive is: if n is not divisible by 3, then n^2 is not divisible by 3. If n is not divisible by 3, then $n = 3k+1$ or $n = 3k+2$ for some $k \in \mathbb{Z}$.
If $n = 3k+1$: $n^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$, remainder 1.
If $n = 3k+2$: $n^2 = 9k^2+12k+4 = 3(3k^2+4k+1)+1$, remainder 1.
In both cases n^2 is not divisible by 3. Since the contrapositive is true, the original statement is true. ∎

5. Completing the square: $x^2+6x+10 = (x+3)^2+1$. Since $(x+3)^2 \geq 0$ for all real x, we have $(x+3)^2+1 \geq 1$. Hence $x^2+6x+10 \geq 1$ for all real x. ∎

6. Suppose, for contradiction, that a largest negative rational number r exists. Then $\frac{r}{2}$ is also rational, and since $r < 0$, dividing by 2 gives a number closer to zero, so $r < \frac{r}{2} < 0$. So $\frac{r}{2}$ is a negative rational number greater than r, contradicting the assumption that r is largest. Hence no largest negative rational number exists. ∎

7. Suppose, for contradiction, that $\sqrt{5}$ is rational, so $\sqrt{5} = \frac{p}{q}$ where $p,q \in \mathbb{Z}$ share no common factor and $q \neq 0$. Then $p^2 = 5q^2$, so $5 \mid p^2$, and since 5 is prime, $5 \mid p$. Write $p = 5k$. Then $25k^2 = 5q^2 \Rightarrow 5k^2 = q^2$, so $5 \mid q^2$, and hence $5 \mid q$. But then 5 divides both p and q, contradicting that they share no common factor. Hence $\sqrt{5}$ is irrational. ∎

8. Suppose, for contradiction, that p and q are not both even, i.e. at least one is odd.
Case 1 (both odd): $p = 2a+1$, $q = 2b+1$. Then $p^2+q^2 = 4a^2+4a+4b^2+4b+2 = 4(a^2+a+b^2+b)+2$, remainder 2 on division by 4.
Case 2 (opposite parity): without loss of generality $p = 2a$, $q = 2b+1$. Then $p^2+q^2 = 4a^2+4b^2+4b+1 = 4(a^2+b^2+b)+1$, remainder 1 on division by 4.
In both cases p^2+q^2 is not exactly divisible by 4, contradicting the given hypothesis. Hence p and q must both be even. ∎

9. Suppose, for contradiction, that $x = k$ is an integer root of $5x^3+10x+3 = 0$. Then $5k^3+10k = -3$, i.e. $5(k^3+2k) = -3$. The left-hand side is divisible by 5 (since $k^3+2k \in \mathbb{Z}$), but -3 is not divisible by 5. This is a contradiction. Hence the equation has no integer roots. ∎

10. (a) Since $\log_3 2 > 0$, we may assume $\log_3 2 = \frac{p}{q}$ for some positive integers p,q. Then $3^{p/q} = 2$, and raising both sides to the power q gives $3^p = 2^q$. (b) Since $p \geq 1$, 3^p is a product of odd numbers and so is odd. Since $q \geq 1$, 2^q is even. But $3^p = 2^q$ would then equate an odd number to an even number, which is impossible. This contradiction shows that $\log_3 2$ cannot be rational; hence $\log_3 2$ is irrational. ∎