Mathematical Proof

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Prove directly that the sum of any two even integers is even. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Prove directly that the product of any two odd integers is odd. **[4]**

3. [Maximum mark: 4] *Difficulty:* 2/5

Disprove the following statement by giving a counterexample:

"For all integers $n \geq 1$, $n^2 + n + 1$ is prime."

[4]

4. [Maximum mark: 5] *Difficulty:* 2/5

Prove by contrapositive that, for $n \in \mathbb{Z}$, if n^2 is even then n is even. **[5]**

5. [Maximum mark: 5] *Difficulty:* 2/5

Prove directly that $x^2 - 4x + 5 > 0$ for all real values of x. **[5]**

6. [Maximum mark: 6] *Difficulty:* 3/5

Prove by contradiction that there is no smallest positive rational number. **[6]**

7. [Maximum mark: 6] *Difficulty:* 3/5

Prove by contradiction that $\sqrt{3}$ is irrational. **[6]**

8. [Maximum mark: 6] *Difficulty:* 3/5

Consider integers m and n such that $m^2 + n^2$ is exactly divisible by 4. Prove by contradiction that m and n cannot have opposite parity (i.e. one cannot be even while the other is odd). **[6]**

9. [Maximum mark: 7] *Difficulty:* 4/5

Prove by contradiction that the equation $3x^3 + 9x + 2 = 0$ has no integer roots. **[7]**

10. [Maximum mark: 9] *Difficulty:* 5/5

Challenge question. Prove by contradiction that $\log_2 3$ is irrational.

(a) Assume $\log_2 3$ is rational and equal to $\dfrac{p}{q}$ for some positive integers p, q. Show that this leads to $2^p = 3^q$. **[4]**

(b) By considering the parity of 2^p and 3^q, complete the proof. **[5]**

Answer Key

1. Let $a = 2m$ and $b = 2n$ for some $m,n \in \mathbb{Z}$. Then $a+b = 2m+2n = 2(m+n)$, which is even since $m+n \in \mathbb{Z}$. Hence the sum of two even integers is even. ■

2. Let $a = 2m+1$ and $b = 2n+1$ for some $m,n \in \mathbb{Z}$. Then $ab = (2m+1)(2n+1) = 4mn+2m+2n+1 = 2(2mn+m+n)+1$, which is odd since $2mn+m+n \in \mathbb{Z}$. Hence the product of two odd integers is odd. ■

3. Let $n = 4$: $n^2+n+1 = 16+4+1 = 21 = 3\times 7$, which is not prime. This single counterexample disproves the statement.

4. The contrapositive is: if n is odd, then n^2 is odd. Let $n = 2k+1$; then $n^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$, which is odd. Since the contrapositive is true, the original statement is true. ■

5. Completing the square: $x^2-4x+5 = (x-2)^2+1$. Since $(x-2)^2 \geq 0$ for all real x, we have $(x-2)^2+1 \geq 1 > 0$. Hence $x^2-4x+5 > 0$ for all real x. ■

6. Suppose, for contradiction, that a smallest positive rational number r exists. Then $\frac{r}{2}$ is also rational, and since $r > 0$, we have $0 < \frac{r}{2} < r$. So $\frac{r}{2}$ is a positive rational number smaller than r, contradicting the assumption that r is smallest. Hence no smallest positive rational number exists. ■

7. Suppose, for contradiction, that $\sqrt{3}$ is rational, so $\sqrt{3} = \frac{p}{q}$ where $p,q \in \mathbb{Z}$ share no common factor and $q \neq 0$. Then $p^2 = 3q^2$, so $3 \mid p^2$, and since 3 is prime, $3 \mid p$. Write $p = 3k$. Then $9k^2 = 3q^2 \Rightarrow 3k^2 = q^2$, so $3 \mid q^2$, and hence $3 \mid q$. But then 3 divides both p and q, contradicting that they share no common factor. Hence $\sqrt{3}$ is irrational. ■

8. Suppose, for contradiction, that m and n have opposite parity; without loss of generality let $m = 2a$ (even) and $n = 2b+1$ (odd) for some $a,b \in \mathbb{Z}$. Then

$$m^2+n^2 = 4a^2+(2b+1)^2 = 4a^2+4b^2+4b+1 = 4(a^2+b^2+b)+1,$$

which leaves remainder 1 on division by 4, contradicting the assumption that m^2+n^2 is exactly divisible by 4. Hence m and n cannot have opposite parity. ■

9. Suppose, for contradiction, that $x = k$ is an integer root of $3x^3+9x+2 = 0$. Then $3k^3+9k = -2$, i.e. $3(k^3+3k) = -2$. The left-hand side is divisible by 3 (since $k^3+3k \in \mathbb{Z}$), but -2 is not divisible by 3. This is a contradiction. Hence the equation has no integer roots. ■

10. (a) Since $\log_2 3 > 0$, we may assume $\log_2 3 = \frac{p}{q}$ for some positive integers p,q. Then $2^{p/q} = 3$, and raising both sides to the power q gives $2^p = 3^q$. (b) Since $p \geq 1$, 2^p is even. Since $q \geq 1$, 3^q is a product of odd numbers and so is odd. But $2^p = 3^q$ would then equate an even number to an odd number, which is impossible. This contradiction shows that $\log_2 3$ cannot be rational; hence $\log_2 3$ is irrational. ■