Maclaurin Series

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 5] *Difficulty:* 1/5

Using the known Maclaurin series for $\cos x$, find the Maclaurin series for $f(x) = \cos(4x)$ up to and including the x^4 term. **[5]**

2. [Maximum mark: 6] *Difficulty:* 1/5

By finding successive derivatives, find the Maclaurin series for $f(x) = e^{-2x}$ up to and including the x^3 term. **[6]**

3. [Maximum mark: 6] *Difficulty:* 2/5

Using the known Maclaurin series for $\sin x$, find the Maclaurin series for $f(x) = x\sin(3x)$ up to and including the x^4 term. **[6]**

4. [Maximum mark: 7] *Difficulty:* 2/5

Using the known Maclaurin series for $\cos x$ and for $\ln(1+x)$, find the Maclaurin series for $f(x) = \cos(x)\ln(1+x)$ up to and including the x^3 term. **[7]**

5. [Maximum mark: 5] *Difficulty:* 2/5

By using the Maclaurin series for $\sin x$, find the value of $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$. **[5]**

6. [Maximum mark: 8] *Difficulty:* 3/5

(a) Find the Maclaurin series for $f(x) = e^{-x^2}$ up to and including the x^8 term. **[4]**

(b) Hence find an approximate value for $\int_0^{0.4} e^{-x^2}\,dx$, giving your answer correct to five decimal places. **[4]**

7. [Maximum mark: 9] *Difficulty:* 3/5

Let $f(x) = \dfrac{1}{2-x}$, for $x \neq 2$.

(a) Show that $f'(x) = \dfrac{1}{(2-x)^2}$. **[2]**

(b) Use mathematical induction to prove that $f^{(n)}(x) = \dfrac{n!}{(2-x)^{n+1}}$ for $n \in \mathbb{Z}^+$. **[7]**

8. [Maximum mark: 9] *Difficulty:* 4/5

Let $g(x) = e^{-2x}\sin x$, for $x \in \mathbb{R}$.

(a) Show that $g''(x) = -4g'(x) - 5g(x)$. **[4]**

(b) Hence find the Maclaurin series for $g(x)$ up to and including the x^4 term. **[5]**

9. [Maximum mark: 7] *Difficulty:* 4/5

Let $h(x) = \cos(3x) \times e^{kx}$, where $k \in \mathbb{Q}$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of 0.

Find the possible values of k. **[7]**

10. [Maximum mark: 20] *Difficulty:* 5/5

Challenge question. Let $f(x) = \ln(1+2x)$, for $x > -\dfrac{1}{2}$.

(a) Show that $f''(x) = -\dfrac{4}{(1+2x)^2}$. **[3]**

(b) Use mathematical induction to prove that

$$f^{(n)}(x) = (-1)^{n-1}\frac{2^n(n-1)!}{(1+2x)^n}$$

for $n \in \mathbb{Z}^+$. **[9]**

Let $g(x) = e^{mx}$, where $m \in \mathbb{Q}$. Consider the function h defined by $h(x) = f(x) \times g(x)$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $-\dfrac{1}{2}$.

(c) Find the value of m. Explain why, unlike in similar problems, only a single value of m is obtained. **[8]**

Answer Key

1. $\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} - \ldots$; substituting $u = 4x$: $\cos(4x) = 1 - 8x^2 + \frac{32}{3}x^4 - \ldots$

2. $f(x) = e^{-2x}$: $f(0) = 1$, $f'(0) = -2$, $f''(0) = 4$, $f'''(0) = -8$. So $e^{-2x} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots$

3. $\sin u = u - \frac{u^3}{6} + \ldots$; substituting $u = 3x$: $\sin(3x) = 3x - \frac{9}{2}x^3 + \ldots$; multiplying by x: $x\sin(3x) = 3x^2 - \frac{9}{2}x^4 + \ldots$

4. $\cos x = 1 - \frac{x^2}{2} + \ldots$; $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$. Multiplying and collecting terms up to x^3: $\cos(x)\ln(1+x) = x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots$

5. $\sin x - x = -\frac{x^3}{6} + \frac{x^5}{120} - \ldots$, so $\frac{\sin x - x}{x^3} = -\frac{1}{6} + \frac{x^2}{120} - \cdots \to -\frac{1}{6}$ as $x \to 0$.

6. (a) $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \ldots$ (b) $\int_0^{0.4} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right) dx \approx 0.37965$.

7. (a) $f'(x) = \frac{d}{dx}(2-x)^{-1} = (2-x)^{-2} = \frac{1}{(2-x)^2}$. (b) *Base case* ($n = 1$): $f'(x) = (2-x)^{-2} = \frac{1!}{(2-x)^2}$, true. *Inductive step*: assume $f^{(k)}(x) = \frac{k!}{(2-x)^{k+1}}$. Differentiating: $f^{(k+1)}(x) = k! \times (k+1)(2-x)^{-k-2} = \frac{(k+1)!}{(2-x)^{k+2}}$, matching the required form with $n = k+1$. By induction, the formula holds for all $n \in \mathbb{Z}^+$. ■

8. (a) $g'(x) = -2e^{-2x}\sin x + e^{-2x}\cos x = -2g(x) + e^{-2x}\cos x$. $g''(x) = 3e^{-2x}\sin x - 4e^{-2x}\cos x$. Since $e^{-2x}\cos x = g'(x) + 2g(x)$ and $e^{-2x}\sin x = g(x)$: $g''(x) = 3g(x) - 4(g'(x) + 2g(x)) = -4g'(x) - 5g(x)$. (b) $g(0) = 0$, $g'(0) = 1$, $g''(0) = -4(1) - 5(0) = -4$. $g'''(x) = -4g''(x) - 5g'(x) \Rightarrow g'''(0) = -4(-4) - 5(1) = 11$. $g^{(4)}(x) = -4g'''(x) - 5g''(x) \Rightarrow g^{(4)}(0) = -4(11) - 5(-4) = -24$. So $g(x) = x - 2x^2 + \frac{11}{6}x^3 - x^4 + \ldots$

9. $\cos(3x) = 1 - \frac{9}{2}x^2 + \ldots$; $e^{kx} = 1 + kx + \frac{k^2x^2}{2} + \ldots$. Product coefficient of x^2: $\frac{k^2}{2} - \frac{9}{2} = 0 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$.

10. (a) $f'(x) = \frac{2}{1+2x}$; $f''(x) = 2 \times (-2)(1+2x)^{-2} = -\frac{4}{(1+2x)^2}$. (b) *Base case* ($n = 1$): $f'(x) = 2(1+2x)^{-1}$, and $(-1)^0 \frac{2^1\,0!}{1+2x} = \frac{2}{1+2x}$, matching. *Inductive step*: assume $f^{(k)}(x) = (-1)^{k-1}\frac{2^k(k-1)!}{(1+2x)^k}$. Differentiating: $f^{(k+1)}(x) = (-1)^{k-1}2^k(k-1)! \times (-2k)(1+2x)^{-k-1} = (-1)^k \frac{2^{k+1}k!}{(1+2x)^{k+1}}$, matching the required form with $n = k+1$. By induction, the formula holds for all $n \in \mathbb{Z}^+$. ■ (c) $f(x) = 2x - 2x^2 + \ldots$ (note $f(0) = 0$); $g(x) = 1 + mx + \frac{m^2x^2}{2} + \ldots$. Product coefficient of x^2: $f(0) \cdot \frac{m^2}{2} + f'(0) \cdot m + f''(0) \cdot \frac{1}{2} = 0 + 2m + (-2) = 2m - 2$. Setting $2m - 2 = -\frac{1}{2} \Rightarrow m = \frac{3}{4}$. Only one value arises (rather than the usual two) because $f(0) = 0$: this eliminates the m^2 term from the coefficient equation entirely, leaving a linear (not quadratic) equation in m.