Maclaurin Series

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

---

1. [Maximum mark: 5] *Difficulty:* 1/5

Using the known Maclaurin series for $\sin x$, find the Maclaurin series for $f(x) = \sin(2x)$ up to and including the x^5 term. **[5]**

2. [Maximum mark: 6] *Difficulty:* 1/5

By finding successive derivatives, find the Maclaurin series for $f(x) = e^{3x}$ up to and including the x^3 term. **[6]**

3. [Maximum mark: 6] *Difficulty:* 2/5

Using the known Maclaurin series for $\cos x$, find the Maclaurin series for $f(x) = x\cos(2x)$ up to and including the x^4 term. **[6]**

4. [Maximum mark: 7] *Difficulty:* 2/5

Using the known Maclaurin series for $\sin x$ and for $\ln(1+x)$, find the Maclaurin series for $f(x) = \sin(x)\ln(1+x)$ up to and including the x^3 term. **[7]**

5. [Maximum mark: 5] *Difficulty:* 2/5

By using the Maclaurin series for $\cos x$, find the value of $\lim_{x\to 0} \dfrac{\cos x - 1}{x^2}$. **[5]**

6. [Maximum mark: 8] *Difficulty:* 3/5

(a) Find the Maclaurin series for $f(x) = \cos(x^2)$ up to and including the x^8 term. **[4]**

(b) Hence find an approximate value for $\displaystyle\int_0^{0.6} \cos(x^2)\,dx$, giving your answer correct to five decimal places. **[4]**

7. [Maximum mark: 9] *Difficulty:* 3/5

Let $f(x) = \dfrac{1}{1+x}$, for $x \neq -1$.

(a) Show that $f'(x) = -\dfrac{1}{(1+x)^2}$. **[2]**

(b) Use mathematical induction to prove that $f^{(n)}(x) = (-1)^n \dfrac{n!}{(1+x)^{n+1}}$ for $n \in \mathbb{Z}^+$. **[7]**

8. [Maximum mark: 9] *Difficulty: 4/5*

Let $g(x) = e^{2x} \cos x$, for $x \in \mathbb{R}$.

(a) Show that $g''(x) = 4g'(x) - 5g(x)$. **[4]**

(b) Hence find the Maclaurin series for $g(x)$ up to and including the x^4 term. **[5]**

9. [Maximum mark: 7] *Difficulty: 4/5*

Let $h(x) = \cos(2x) \times e^{kx}$, where $k \in \mathbb{Q}$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\dfrac{1}{2}$.

Find the possible values of k. **[7]**

10. [Maximum mark: 20] *Difficulty: 5/5*

Challenge question. Let $f(x) = (1+x)^{-3/2}$, for $x > -1$.

(a) Show that $f''(x) = \dfrac{15}{4}(1+x)^{-7/2}$. **[3]**

(b) Use mathematical induction to prove that

$$f^{(n)}(x) = (-1)^n \frac{(2n+1)!}{4^n \, n!} (1+x)^{-(2n+3)/2}$$

for $n \in \mathbb{Z}^+$. **[9]**

Let $g(x) = e^{mx}$, where $m \in \mathbb{Q}$. Consider the function h defined by $h(x) = f(x) \times g(x)$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\dfrac{19}{8}$.

(c) Find the possible values of m, giving your answers in exact form. **[8]**

Answer Key

1. $\sin u = u - \frac{u^3}{6} + \frac{u^5}{120} - \dots$; substituting $u = 2x$: $\sin(2x) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots$

2. $f(x) = e^{3x}$: $f(0) = 1$, $f'(0) = 3$, $f''(0) = 9$, $f'''(0) = 27$. So $e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots$

3. $\cos u = 1 - \frac{u^2}{2} + \dots$; substituting $u = 2x$: $\cos(2x) = 1 - 2x^2 + \dots$; multiplying by x: $x\cos(2x) = x - 2x^3 + \dots$

4. $\sin x = x - \frac{x^3}{6} + \dots$; $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$. Multiplying and collecting terms up to x^3: $\sin(x)\ln(1+x) = x^2 - \frac{1}{2}x^3 + \dots$

5. $\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} - \dots$, so $\frac{\cos x - 1}{x^2} = -\frac{1}{2} + \frac{x^2}{24} - \dots \to -\frac{1}{2}$ as $x \to 0$.

6. (a) $\cos(x^2) = 1 - \frac{x^4}{2} + \frac{x^8}{24} - \dots$ (b) $\int_0^{0.6} \left(1 - \frac{x^4}{2} + \frac{x^8}{24}\right) dx = \left[x - \frac{x^5}{10} + \frac{x^9}{216}\right]_0^{0.6} \approx 0.59227$.

7. (a) $f'(x) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$. (b) *Base case* ($n = 1$): $f'(x) = -(1+x)^{-2} = (-1)^1\frac{1!}{(1+x)^2}$, true. *Inductive step*: assume $f^{(k)}(x) = (-1)^k\frac{k!}{(1+x)^{k+1}}$. Differentiating: $f^{(k+1)}(x) = (-1)^k k! \times (-(k+1))(1+x)^{-k-2} = (-1)^{k+1}\frac{(k+1)!}{(1+x)^{k+2}}$, matching the required form with $n = k+1$. By induction, the formula holds for all $n \in \mathbb{Z}^+$. ∎

8. (a) $g'(x) = -e^{2x}\sin x + 2e^{2x}\cos x = 2g(x) - e^{2x}\sin x$. $g''(x) = -4e^{2x}\sin x + 3e^{2x}\cos x$. Since $e^{2x}\sin x = 2g(x) - g'(x)$ and $e^{2x}\cos x = g(x)$: $g''(x) = -4(2g(x) - g'(x)) + 3g(x) = 4g'(x) - 5g(x)$. (b) $g(0) = 1$, $g'(0) = 2$, $g''(0) = 4(2) - 5(1) = 3$. $g'''(x) = 4g''(x) - 5g'(x) \Rightarrow g'''(0) = 4(3) - 5(2) = 2$. $g^{(4)}(x) = 4g'''(x) - 5g''(x) \Rightarrow g^{(4)}(0) = 4(2) - 5(3) = -7$. So $g(x) = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 + \dots$

9. $\cos(2x) = 1 - 2x^2 + \dots$; $e^{kx} = 1 + kx + \frac{k^2x^2}{2} + \dots$. Product coefficient of x^2: $\frac{k^2}{2} - 2 = \frac{1}{2} \Rightarrow k^2 = 5 \Rightarrow k = \pm\sqrt{5}$.

10. (a) $f'(x) = -\frac{3}{2}(1+x)^{-5/2}$; $f''(x) = -\frac{3}{2} \times \left(-\frac{5}{2}\right)(1+x)^{-7/2} = \frac{15}{4}(1+x)^{-7/2}$. (b) *Base case* ($n = 1$): $f'(x) = -\frac{3}{2}(1+x)^{-5/2}$, and $(-1)^1\frac{3!}{4^1\,1!} = -\frac{6}{4} = -\frac{3}{2}$, matching. *Inductive step*: assume $f^{(k)}(x) = (-1)^k\frac{(2k+1)!}{4^k\,k!}(1+x)^{-(2k+3)/2}$. Differentiating:

$$f^{(k+1)}(x) = (-1)^k\frac{(2k+1)!}{4^k\,k!} \times \left(-\frac{2k+3}{2}\right)(1+x)^{-(2k+5)/2} = (-1)^{k+1}\frac{(2k+1)!(2k+3)}{4^{k+1}(k+1)!}(1+x)^{-(2(k+1)+3)/2}.$$

Since $(2k+1)!(2k+3) = (2k+3)!/(2k+2)$ and $(k+1)! = (k+1)k!$, simplifying using $(2k+3)! = (2k+3)(2k+2)(2k+1)!$ confirms this equals $(-1)^{k+1}\frac{(2(k+1)+1)!}{4^{k+1}(k+1)!}(1+x)^{-(2(k+1)+3)/2}$, matching the required form with $n = k+1$. By induction, the formula holds for all $n \in \mathbb{Z}^+$. ∎ (c) $f(x) = 1 - \frac{3x}{2} + \frac{15x^2}{8} - \dots$; $g(x) = 1 + mx + \frac{m^2x^2}{2} + \dots$. Product coefficient of x^2: $\frac{m^2}{2} - \frac{3m}{2} + \frac{15}{8} = \frac{19}{8} \Rightarrow 4m^2 - 12m - 4 = 0 \Rightarrow m^2 - 3m - 1 = 0 \Rightarrow m = \frac{3 \pm \sqrt{13}}{2}$.