Maclaurin Series

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 5] *Difficulty:* 1/5

Using the known Maclaurin series for $\cos x$, find the Maclaurin series for $f(x) = \cos(3x)$ up to and including the x^4 term. **[5]**

2. [Maximum mark: 6] *Difficulty:* 1/5

By finding successive derivatives, find the Maclaurin series for $f(x) = e^{2x}$ up to and including the x^3 term. **[6]**

3. [Maximum mark: 6] *Difficulty:* 2/5

Using the known Maclaurin series for $\sin x$, find the Maclaurin series for $f(x) = x\sin(2x)$ up to and including the x^4 term. **[6]**

4. [Maximum mark: 7] *Difficulty:* 2/5

Using the known Maclaurin series for e^x and for $\ln(1+x)$, find the Maclaurin series for $f(x) = e^x \ln(1+x)$ up to and including the x^3 term. **[7]**

5. [Maximum mark: 5] *Difficulty:* 2/5

By using the Maclaurin series for e^x, find the value of $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$. **[5]**

6. [Maximum mark: 8] *Difficulty:* 3/5

(a) Find the Maclaurin series for $f(x) = \sin(x^2)$ up to and including the x^{10} term. **[4]**

(b) Hence find an approximate value for $\int_0^{0.5} \sin(x^2)\,dx$, giving your answer correct to five decimal places. **[4]**

7. [Maximum mark: 9] *Difficulty:* 3/5

Let $f(x) = \ln(1+x)$, for $x > -1$.

(a) Show that $f'(x) = \dfrac{1}{1+x}$. **[2]**

(b) Use mathematical induction to prove that $f^{(n)}(x) = (-1)^{n-1}\dfrac{(n-1)!}{(1+x)^n}$ for $n \in \mathbb{Z}^+$. **[7]**

8. [Maximum mark: 9] *Difficulty:* 4/5

Let $g(x) = e^{-x}\sin x$, for $x \in \mathbb{R}$.

(a) Show that $g''(x) = -2g'(x) - 2g(x)$. **[4]**

(b) Hence find the Maclaurin series for $g(x)$ up to and including the x^4 term. **[5]**

9. [Maximum mark: 7] *Difficulty:* 4/5

Let $h(x) = \cos(x) \times e^{kx}$, where $k \in \mathbb{Q}$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\dfrac{3}{2}$.

Find the possible values of k. **[7]**

10. [Maximum mark: 20] *Difficulty:* 5/5

Challenge question. Let $f(x) = (1-x)^{-1/2}$, for $x < 1$.

(a) Show that $f''(x) = \dfrac{3}{4}(1-x)^{-5/2}$. **[3]**

(b) Use mathematical induction to prove that

$$f^{(n)}(x) = \frac{(2n)!}{4^n\, n!}(1-x)^{-(2n+1)/2}$$

for $n \in \mathbb{Z}^+$. **[9]**

Let $g(x) = e^{mx}$, where $m \in \mathbb{Q}$. Consider the function h defined by $h(x) = f(x) \times g(x)$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\dfrac{11}{8}$.

(c) Find the possible values of m. **[8]**

Answer Key

1. $\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \dots$; substituting $u = 3x$: $\cos(3x) = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \dots$

2. $f(x) = e^{2x}$: $f(0) = 1$, $f'(0) = 2$, $f''(0) = 4$, $f'''(0) = 8$. So $e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$

3. $\sin u = u - \frac{u^3}{6} + \dots$; substituting $u = 2x$: $\sin(2x) = 2x - \frac{4x^3}{3} + \dots$; multiplying by x: $x\sin(2x) = 2x^2 - \frac{4}{3}x^4 + \dots$

4. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$; $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$. Multiplying and collecting terms up to x^3: $e^x \ln(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

5. $e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{6} + \dots$, so $\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{6} + \dots \to \frac{1}{2}$ as $x \to 0$.

6. (a) $\sin(x^2) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \dots$ (b) $\int_0^{0.5} \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}\right) dx = \left[\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320}\right]_0^{0.5} \approx 0.04148$.

7. (a) $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$. (b) *Base case* ($n = 1$): $f'(x) = (1+x)^{-1} = (-1)^0 \frac{0!}{(1+x)^1}$, true. *Inductive step*: assume $f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k}$. Differentiating: $f^{(k+1)}(x) = (-1)^{k-1}(k-1)! \times (-k)(1+x)^{-k-1} = (-1)^k \frac{k!}{(1+x)^{k+1}}$, which matches the required form with $n = k+1$. Since true for $n = 1$ and the truth of $n = k$ implies truth of $n = k+1$, by induction the formula holds for all $n \in \mathbb{Z}^+$. ■

8. (a) $g'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}\cos x - g(x)$. $g''(x) = -2e^{-x}\cos x$. Since $e^{-x}\cos x = g'(x) + g(x)$: $g''(x) = -2(g'(x) + g(x)) = -2g'(x) - 2g(x)$. (b) $g(0) = 0$, $g'(0) = 1$, $g''(0) = -2g'(0) - 2g(0) = -2$. $g'''(x) = -2g''(x) - 2g'(x) \Rightarrow g'''(0) = -2(-2) - 2(1) = 2$. $g^{(4)}(x) = -2g'''(x) - 2g''(x) \Rightarrow g^{(4)}(0) = -2(2) - 2(-2) = 0$. So $g(x) = x - x^2 + \frac{x^3}{3} + 0 \cdot x^4 + \dots = x - x^2 + \frac{x^3}{3} + \dots$

9. $\cos x = 1 - \frac{x^2}{2} + \dots$; $e^{kx} = 1 + kx + \frac{k^2x^2}{2} + \dots$. Product coefficient of x^2: $\frac{k^2}{2} - \frac{1}{2} = \frac{3}{2} \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$.

10. (a) $f'(x) = \frac{1}{2}(1-x)^{-3/2}$; $f''(x) = \frac{1}{2} \times \frac{3}{2}(1-x)^{-5/2} = \frac{3}{4}(1-x)^{-5/2}$. (b) *Base case* ($n = 1$): $f'(x) = \frac{1}{2}(1-x)^{-3/2}$, and $\frac{2!}{4^1\,1!} = \frac{2}{4} = \frac{1}{2}$, matching. *Inductive step*: assume $f^{(k)}(x) = \frac{(2k)!}{4^k\,k!}(1-x)^{-(2k+1)/2}$. Differentiating: $f^{(k+1)}(x) = \frac{(2k)!}{4^k\,k!} \times \frac{2k+1}{2}(1-x)^{-(2k+3)/2} = \frac{(2k)!(2k+1)}{4^k\,k! \times 2}(1-x)^{-(2k+3)/2}$. Since $(2k)!(2k+1) = (2k+1)!$ and $4^k \times 2 = \frac{4^{k+1}}{2}$, and $(2k+2) = 2(k+1)$: rewriting, $f^{(k+1)}(x) = \frac{(2k+2)!}{4^{k+1}(k+1)!}(1-x)^{-(2(k+1)+1)/2}$, matching the required form with $n = k+1$. By induction, the formula holds for all $n \in \mathbb{Z}^+$. ■ (c) $f(x) = 1 + \frac{x}{2} + \frac{3x^2}{8} + \dots$; $g(x) = 1 + mx + \frac{m^2x^2}{2} + \dots$. Product coefficient of x^2: $\frac{m^2}{2} + \frac{m}{2} + \frac{3}{8} = \frac{11}{8} \Rightarrow 4m^2 + 4m - 8 = 0 \Rightarrow m^2 + m - 2 = 0 \Rightarrow (m+2)(m-1) = 0 \Rightarrow m = 1$ or $m = -2$.