Integration

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 5] *Difficulty:* 1/5

Find the value of $\int_1^{16} \left(\frac{5\sqrt{x}-4}{\sqrt{x}} \right) dx$. **[5]**

2. [Maximum mark: 5] *Difficulty:* 1/5

The derivative of a function g is given by $g'(x) = 9x^2 + 2e^x$, where $x \in \mathbb{R}$. The graph of g passes through the point $(0,6)$. Find $g(x)$. **[5]**

3. [Maximum mark: 5] *Difficulty:* 2/5

The derivative of the function f is given by $f'(x) = \frac{12x}{x^2+5}$.

The graph of $y = f(x)$ passes through the point $(2,10)$. Find an expression for $f(x)$. **[5]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Find the value of $\int_0^1 x(x^2+2)^3\, dx$. **[6]**

5. [Maximum mark: 7] *Difficulty:* 2/5

Find the value of $\int_0^1 xe^{4x}\, dx$. **[7]**

6. [Maximum mark: 6] *Difficulty:* 3/5

By using the substitution $u = \cos x$, or otherwise, find an expression for $\int_0^{\pi/3} \cos^n x \sin x\, dx$ in terms of n, where n is a non-zero real number. **[6]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Find the area of the region enclosed by the curve $y = x^2$ and the line $y = 5x - 4$. **[7]**

8. [Maximum mark: 7] *Difficulty:* 4/5

The region bounded by the curve $y = \sqrt{3x+2}$, the x-axis, and the lines $x = 0$ and $x = 5$ is rotated 360° about the x-axis.

Find the exact volume of the solid formed. **[7]**

9. [Maximum mark: 8] *Difficulty:* 4/5

The gradient of a curve is given by $\frac{dy}{dx} = 2xy$, where $y \neq 0$. The curve passes through the point $(0,3)$.

Find an expression for y in terms of x. **[8]**

10. [Maximum mark: 16] *Difficulty: 5/5*

Challenge question. The following diagram shows the curve $\frac{x^2}{36} + \frac{(y-7)^2}{49} = 1$, where $0 \leq y \leq 14$ and $x \geq 0$.

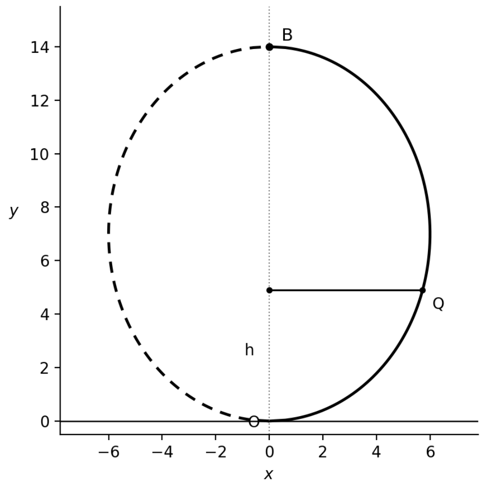

diagram not to scale

The curve from point Q to point B is rotated 360° about the y-axis to form the interior surface of a bowl, where Q lies at height $y = h$ on the curve and $B = (0,14)$ is the rim of the bowl, with $0 < h < 14$.

Given that the interior volume of the bowl is 600 cm^3, determine the value of h. **[16]**

Answer Key

1. $\dfrac{5\sqrt{x}-4}{\sqrt{x}} = 5-4x^{-1/2}$. $\int (5-4x^{-1/2})\,dx = 5x - 8x^{1/2} + C$. Evaluating: at $x=16$: $80-32=48$; at $x=1$: $5-8=-3$. Value $= 48-(-3)=51$.

2. $g(x) = 3x^3+2e^x+C$. $g(0)=0+2+C=6 \Rightarrow C=4$. So $g(x)=3x^3+2e^x+4$.

3. Since $\dfrac{d}{dx}\ln(x^2+5) = \dfrac{2x}{x^2+5}$, we have $f(x) = 6\ln(x^2+5)+C$. $f(2)=6\ln 9 + C = 10 \Rightarrow C = 10-6\ln 9$. So $f(x) = 6\ln(x^2+5)+10-6\ln 9 = 6\ln\left(\dfrac{x^2+5}{9}\right)+10$.

4. Let $u=x^2+2$, $du = 2x\,dx$. Limits: $x=0 \Rightarrow u=2$; $x=1 \Rightarrow u=3$. $\int_2^3 u^3\cdot\frac{1}{2}\,du = \frac{1}{2}\left[\dfrac{u^4}{4}\right]_2^3 = \frac{1}{8}(81-16) = \dfrac{65}{8}$.

5. Using integration by parts with $u=x$, $dv=e^{4x}dx$: $\int xe^{4x}dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}+C$. Evaluating from 0 to 1: $\left(\frac{1}{4}e^4 - \frac{1}{16}e^4\right) - \left(0-\frac{1}{16}\right) = \frac{3}{16}e^4 + \frac{1}{16} = \dfrac{3e^4+1}{16}$.

6. Let $u=\cos x$, $du = -\sin x\,dx$. Limits: $x=0\Rightarrow u=1$; $x=\frac{\pi}{3} \Rightarrow u=\frac{1}{2}$. $\int_0^{\pi/3}\cos^n x\sin x\,dx = -\int_1^{1/2} u^n\,du = \int_{1/2}^1 u^n\,du = \left[\dfrac{u^{n+1}}{n+1}\right]_{1/2}^1 = \dfrac{1-(1/2)^{n+1}}{n+1}$.

7. Intersections: $x^2=5x-4 \Rightarrow x^2-5x+4=0 \Rightarrow (x-1)(x-4)=0 \Rightarrow x=1,4$. Area $= \int_1^4 \left[(5x-4)-x^2\right]dx = \left[\dfrac{5x^2}{2}-4x-\dfrac{x^3}{3}\right]_1^4 = \dfrac{16}{3} - \left(-\dfrac{25}{6}\right) = \dfrac{9}{2}$.

8. $V = \pi\int_0^5 (3x+2)\,dx = \pi\left[\dfrac{3x^2}{2}+2x\right]_0^5 = \pi(37.5+10) = \dfrac{95\pi}{2}$.

9. Separating variables: $\int \dfrac{dy}{y} = \int 2x\,dx \Rightarrow \ln y = x^2+C$. At $(0,3)$: $\ln 3 = 0 + C \Rightarrow C=\ln 3$. So $\ln y = x^2+\ln 3 \Rightarrow y = 3e^{x^2}$.

10. From the curve, $x^2 = 36\left(1-\dfrac{(y-7)^2}{49}\right) = 36 - \dfrac{36}{49}(y-7)^2$. The interior volume is

$$V(h) = \pi\int_h^{14} x^2\,dy = \pi\int_h^{14}\left[36-\frac{36}{49}(y-7)^2\right]dy = \frac{12\pi}{49}\left(h^3-21h^2+1372\right).$$

Setting $V(h)=600$ and solving numerically (GDC) gives three roots, of which only $h\approx 6.36$ lies in the valid interval $0<h<14$ (the other roots fall outside the domain of the curve). So $h \approx 6.36$ cm.