Integration

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 5] *Difficulty:* 1/5

Find the value of $\displaystyle\int_1^9 \left(\frac{4\sqrt{x}-3}{\sqrt{x}}\right) dx$. **[5]**

2. [Maximum mark: 5] *Difficulty:* 1/5

The derivative of a function g is given by $g'(x) = 6x^2 + 4e^x$, where $x \in \mathbb{R}$. The graph of g passes through the point $(0,7)$. Find $g(x)$. **[5]**

3. [Maximum mark: 5] *Difficulty:* 2/5

The derivative of the function f is given by $f'(x) = \dfrac{10x}{x^2+4}$.

The graph of $y = f(x)$ passes through the point $(2,9)$. Find an expression for $f(x)$. **[5]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Find the value of $\displaystyle\int_0^1 x(x^2+3)^2\,dx$. **[6]**

5. [Maximum mark: 7] *Difficulty:* 2/5

Find the value of $\displaystyle\int_0^1 xe^{3x}\,dx$. **[7]**

6. [Maximum mark: 6] *Difficulty:* 3/5

By using the substitution $u = \sin x$, or otherwise, find an expression for $\displaystyle\int_0^{\pi/6} \sin^n x \cos x\,dx$ in terms of n, where n is a non-zero real number. **[6]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Find the area of the region enclosed by the curve $y = x^2$ and the line $y = 4x - 3$. **[7]**

8. [Maximum mark: 7] *Difficulty:* 4/5

The region bounded by the curve $y = \sqrt{2x+1}$, the x-axis, and the lines $x = 0$ and $x = 4$ is rotated 360° about the x-axis.

Find the exact volume of the solid formed. **[7]**

9. [Maximum mark: 8] *Difficulty:* 4/5

The gradient of a curve is given by $\frac{dy}{dx} = x^2 y$, where $y \neq 0$. The curve passes through the point $(0,2)$.

Find an expression for y in terms of x. **[8]**

10. [Maximum mark: 16] *Difficulty: 5/5*

Challenge question. The following diagram shows the curve $\frac{x^2}{16} + \frac{(y-6)^2}{25} = 1$, where $1 \leq y \leq 11$ and $x \geq 0$.

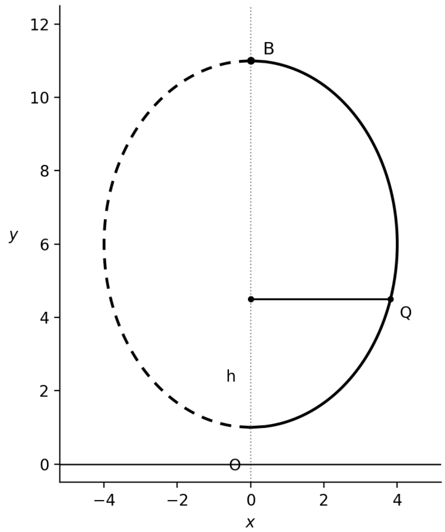

diagram not to scale

The curve from point Q to point B is rotated $360°$ about the y-axis to form the interior surface of a bowl, where Q lies at height $y = h$ on the curve and $B = (0,11)$ is the rim of the bowl, with $1 < h < 11$.

Given that the interior volume of the bowl is $250\ \text{cm}^3$, determine the value of h. **[16]**

Answer Key

1. $\dfrac{4\sqrt{x}-3}{\sqrt{x}} = 4-3x^{-1/2}$. $\int (4-3x^{-1/2})\,dx = 4x-6x^{1/2}+C$. Evaluating: at $x=9$: $36-18=18$; at $x=1$: $4-6=-2$. Value $= 18-(-2) = 20$.

2. $g(x) = 2x^3+4e^x+C$. $g(0) = 0+4+C = 7 \Rightarrow C = 3$. So $g(x) = 2x^3+4e^x+3$.

3. Since $\dfrac{d}{dx}\ln(x^2+4) = \dfrac{2x}{x^2+4}$, we have $f(x) = 5\ln(x^2+4)+C$. $f(2) = 5\ln 8+C = 9 \Rightarrow C = 9-5\ln 8$. So $f(x) = 5\ln(x^2+4)+9-5\ln 8 = 5\ln\left(\dfrac{x^2+4}{8}\right)+9$.

4. Let $u = x^2+3$, $du = 2x\,dx$. Limits: $x=0 \Rightarrow u=3$; $x=1 \Rightarrow u=4$. $\int_3^4 u^2\cdot\frac{1}{2}\,du = \frac{1}{2}\left[\dfrac{u^3}{3}\right]_3^4 = \frac{1}{6}(64-27) = \dfrac{37}{6}$.

5. Using integration by parts with $u = x$, $dv = e^{3x}dx$: $\int xe^{3x}dx = \frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+C$. Evaluating from 0 to 1: $\left(\frac{1}{3}e^3-\frac{1}{9}e^3\right)-\left(0-\frac{1}{9}\right) = \frac{2}{9}e^3+\frac{1}{9} = \dfrac{2e^3+1}{9}$.

6. Let $u = \sin x$, $du = \cos x\,dx$. Limits: $x=0 \Rightarrow u=0$; $x=\frac{\pi}{6} \Rightarrow u=\frac{1}{2}$. $\int_0^{1/2} u^n\,du = \left[\dfrac{u^{n+1}}{n+1}\right]_0^{1/2} = \dfrac{(1/2)^{n+1}}{n+1}$.

7. Intersections: $x^2 = 4x-3 \Rightarrow x^2-4x+3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$. Area $= \int_1^3 \left[(4x-3)-x^2\right]dx = \left[2x^2-3x-\dfrac{x^3}{3}\right]_1^3 = 0-\left(-\dfrac{4}{3}\right) = \dfrac{4}{3}$.

8. $V = \pi\int_0^4 (2x+1)\,dx = \pi\left[x^2+x\right]_0^4 = \pi(16+4) = 20\pi$.

9. Separating variables: $\int\dfrac{dy}{y} = \int x^2\,dx \Rightarrow \ln y = \dfrac{x^3}{3}+C$. At $(0,2)$: $\ln 2 = 0+C \Rightarrow C = \ln 2$. So $\ln y = \dfrac{x^3}{3}+\ln 2 \Rightarrow y = 2e^{x^3/3}$.

10. From the curve, $x^2 = 16\left(1-\dfrac{(y-6)^2}{25}\right) = 16-\dfrac{16}{25}(y-6)^2$. The interior volume is

$$V(h) = \pi\int_h^{11} x^2\,dy = \pi\int_h^{11}\left[16-\frac{16}{25}(y-6)^2\right]dy = \frac{16\pi}{75}\left(h^3-18h^2+33h+484\right).$$

Setting $V(h) = 250$ and solving numerically (GDC) gives three roots, of which only $h \approx 4.29$ lies in the valid interval $1 < h < 11$ (the other roots fall outside the domain of the curve). So $h \approx 4.29$ cm.