Integration

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 5] *Difficulty:* 1/5

Find the value of $\int_1^4 \left(\frac{3\sqrt{x}-2}{\sqrt{x}}\right) dx$. **[5]**

2. [Maximum mark: 5] *Difficulty:* 1/5

The derivative of a function g is given by $g'(x) = 4x^3 + 3e^x$, where $x \in \mathbb{R}$. The graph of g passes through the point $(0,5)$. Find $g(x)$. **[5]**

3. [Maximum mark: 5] *Difficulty:* 2/5

The derivative of the function f is given by $f'(x) = \frac{8x}{x^2+3}$.

The graph of $y = f(x)$ passes through the point $(1,6)$. Find an expression for $f(x)$. **[5]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Find the value of $\int_0^1 x(x^2+1)^3\, dx$. **[6]**

5. [Maximum mark: 7] *Difficulty:* 2/5

Find the value of $\int_0^1 xe^{2x}\, dx$. **[7]**

6. [Maximum mark: 6] *Difficulty:* 3/5

By using the substitution $u = \tan x$, or otherwise, find an expression for $\int_0^{\pi/4} \tan^n x \sec^2 x\, dx$ in terms of n, where n is a non-zero real number. **[6]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Find the area of the region enclosed by the curve $y = x^2$ and the line $y = 2x+3$. **[7]**

8. [Maximum mark: 7] *Difficulty:* 4/5

The region bounded by the curve $y = \sqrt{x+1}$, the x-axis, and the lines $x = 0$ and $x = 3$ is rotated $360°$ about the x-axis.

Find the exact volume of the solid formed. **[7]**

9. [Maximum mark: 8] *Difficulty:* 4/5

The gradient of a curve is given by $\frac{dy}{dx} = xy^2$, where $y \neq 0$. The curve passes through the point $(0, 1)$.

Find an expression for y in terms of x. **[8]**

10. [Maximum mark: 16] *Difficulty: 5/5*

Challenge question. The following diagram shows the curve $\frac{x^2}{25} + \frac{(y-5)^2}{9} = 1$, where $2 \leq y \leq 8$ and $x \geq 0$.

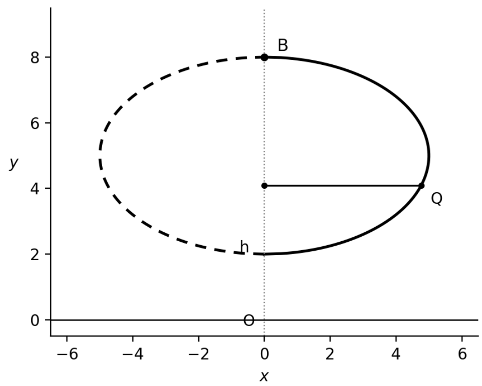

diagram not to scale

The curve from point Q to point B is rotated $360°$ about the y-axis to form the interior surface of a bowl, where Q lies at height $y = h$ on the curve and $B = (0, 8)$ is the rim of the bowl, with $2 < h < 8$.

Given that the interior volume of the bowl is 285 cm^3, determine the value of h. **[16]**

Answer Key

1. $\dfrac{3\sqrt{x}-2}{\sqrt{x}} = 3-2x^{-1/2}$. $\displaystyle\int(3-2x^{-1/2})\,dx = 3x-4x^{1/2}+C$. Evaluating: at $x=4$: $12-8=4$; at $x=1$: $3-4=-1$. Value $=4-(-1)=5$.

2. $g(x)=x^4+3e^x+C$. $g(0)=0+3+C=5 \Rightarrow C=2$. So $g(x)=x^4+3e^x+2$.

3. Since $\dfrac{d}{dx}\ln(x^2+3)=\dfrac{2x}{x^2+3}$, we have $f(x)=4\ln(x^2+3)+C$. $f(1)=4\ln 4+C=6 \Rightarrow C=6-4\ln 4$. So $f(x)=4\ln(x^2+3)+6-4\ln 4=4\ln\left(\dfrac{x^2+3}{4}\right)+6$.

4. Let $u=x^2+1$, $du=2x\,dx$. Limits: $x=0\Rightarrow u=1$; $x=1\Rightarrow u=2$. $\displaystyle\int_1^2 u^3\cdot\tfrac{1}{2}\,du=\tfrac{1}{2}\left[\frac{u^4}{4}\right]_1^2=\tfrac{1}{8}(16-1)=\frac{15}{8}$.

5. Using integration by parts with $u=x$, $dv=e^{2x}dx$: $\displaystyle\int xe^{2x}dx=\tfrac{1}{2}xe^{2x}-\tfrac{1}{4}e^{2x}+C$. Evaluating from 0 to 1: $\left(\tfrac{1}{2}e^2-\tfrac{1}{4}e^2\right)-\left(0-\tfrac{1}{4}\right)=\tfrac{1}{4}e^2+\tfrac{1}{4}=\dfrac{e^2+1}{4}$.

6. Let $u=\tan x$, $du=\sec^2 x\,dx$. Limits: $x=0\Rightarrow u=0$; $x=\frac{\pi}{4}\Rightarrow u=1$. $\displaystyle\int_0^1 u^n\,du=\left[\frac{u^{n+1}}{n+1}\right]_0^1=\frac{1}{n+1}$.

7. Intersections: $x^2=2x+3\Rightarrow x^2-2x-3=0\Rightarrow(x-3)(x+1)=0\Rightarrow x=-1,3$. Area $=\displaystyle\int_{-1}^3\left[(2x+3)-x^2\right]dx=\left[x^2+3x-\frac{x^3}{3}\right]_{-1}^3=9-\left(-\frac{5}{3}\right)=\frac{32}{3}$.

8. $V=\pi\displaystyle\int_0^3(x+1)\,dx=\pi\left[\frac{x^2}{2}+x\right]_0^3=\pi(4.5+3)=7.5\pi$.

9. Separating variables: $\displaystyle\int\frac{dy}{y^2}=\int x\,dx\Rightarrow-\frac{1}{y}=\frac{x^2}{2}+C$. At $(0,1)$: $-1=0+C\Rightarrow C=-1$. So $-\dfrac{1}{y}=\dfrac{x^2}{2}-1\Rightarrow\dfrac{1}{y}=1-\dfrac{x^2}{2}=\dfrac{2-x^2}{2}\Rightarrow y=\dfrac{2}{2-x^2}$.

10. From the curve, $x^2=25\left(1-\dfrac{(y-5)^2}{9}\right)=25-\dfrac{25}{9}(y-5)^2$. The interior volume is

$$V(h)=\pi\int_h^8 x^2\,dy=\pi\int_h^8\left[25-\frac{25}{9}(y-5)^2\right]dy=\frac{25\pi}{27}\left(h^3-15h^2+48h+64\right).$$

Setting $V(h)=285$ and solving numerically (GDC) gives three roots, of which only $h\approx 3.13$ lies in the valid interval $2<h<8$ (the other roots fall outside the domain of the curve). So $h\approx 3.13$ cm.