Geometry

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

(a) Convert 300° to radians, giving your answer as a multiple of π. **[1]**

(b) Convert $\frac{3\pi}{5}$ radians to degrees. **[2]**

2. [Maximum mark: 4] *Difficulty:* 1/5

A circle has radius 12 cm. Find

(a) the length of an arc that subtends an angle of 0.9 radians at the centre; **[2]**

(b) the area of the corresponding sector. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

A circle has centre O and radius 7 m. Points A and B lie on the circle such that $A\hat{O}B = 2.0$ radians.

(a) Find the length of the chord $[AB]$. **[3]**

(b) Find the area of the minor sector OAB. **[2]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Triangle ABC has $AB = 8$ cm, $BC = 12$ cm, and $A\hat{B}C = 70°$.

(a) Find AC. **[3]**

(b) Find the area of triangle ABC. **[3]**

5. [Maximum mark: 7] *Difficulty:* 3/5

Consider a circle with diameter AB, where A has coordinates $(2,1,0)$ and B has coordinates $(-2,1,4)$.

(a) **(i)** Find the coordinates of the centre of the circle. **[2]**

(ii) Find the radius of the circle. **[2]**

The circle forms the base of a right cone whose vertex V has coordinates $(0,8,2)$.

(b) Find the exact volume of the cone. **[3]**

6. [Maximum mark: 6] *Difficulty:* 3/5

A circle has centre O and radius 3 cm. Points C and D lie on the circle such that $C\hat{O}D = \theta$, where $0 < \theta < \pi$.

(a) Find an expression, in terms of θ, for the area of the minor segment cut off by chord $[CD]$. **[3]**

(b) Hence find the area of this segment when $\theta = 1.8$ radians. **[3]**

7. [Maximum mark: 8] *Difficulty:* 4/5

A company designs a logo by removing two equal segments from a rectangle measuring 7 cm by 5 cm. The points A and B lie on a circle, with centre O and radius 2 cm, such that $A\hat{O}B = \theta$, where $0 < \theta < \pi$.

(a) Find the area of one of the shaded segments in terms of θ. **[3]**

(b) Given that the area of the logo is 27.0 cm^2, find the value of θ. **[5]**

8. [Maximum mark: 7] *Difficulty:* 4/5

Consider a quadrilateral $ABCD$ such that $AB = 2$, $BC = 6$, $CD = 8$ and $DA = 10$. Let $\alpha = A\hat{B}C$ and $\beta = A\hat{D}C$.

(a) **(i)** Find AC^2 in terms of α. **[2]**

(ii) Find AC^2 in terms of β. **[2]**

(iii) Hence, or otherwise, find an expression for α in terms of β. **[3]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Using the quadrilateral $ABCD$ from Question 8, find the maximum area of $ABCD$. **[8]**

10. [Maximum mark: 13] *Difficulty:* 5/5

Challenge question. Two sides of a triangle have lengths x cm and $(12 - x)$ cm, where $0 < x < 12$, and the angle between them is $\dfrac{2\pi}{3}$ radians.

(a) Show that the area of the triangle is given by $A(x) = \dfrac{\sqrt{3}}{4}x(12 - x)$. **[3]**

(b) Show that the square of the length of the third side, y cm, is given by $y^2 = x^2 - 12x + 144$. **[4]**

(c) Find the value of x that maximises $A(x)$, and state this maximum area. **[3]**

(d) Determine whether y^2 is also maximised at this same value of x. Justify your answer. **[3]**

Answer Key

1. (a) $300^\circ = \dfrac{5\pi}{3}$. (b) $\dfrac{3\pi}{5} = 108^\circ$.

2. (a) Arc length $= r\theta = 12(0.9) = 10.8$ cm. (b) Sector area $= \dfrac{1}{2}r^2\theta = \dfrac{1}{2}(144)(0.9) = 64.8$ cm^2.

3. (a) $AB^2 = 7^2+7^2-2(7)(7)\cos(2.0) = 98(1-\cos 2.0)$, so $AB \approx 11.8$ m. (b) Sector area $= \dfrac{1}{2}(49)(2.0) = 49$ m^2.

4. (a) $AC^2 = 8^2+12^2-2(8)(12)\cos 70^\circ \approx 142.3$, so $AC \approx 11.9$ cm. (b) Area $= \dfrac{1}{2}(8)(12)\sin 70^\circ \approx 45.1$ cm^2.

5. (a)(i) Centre = midpoint of $AB = (0,1,2)$. (ii) Radius $= \dfrac{1}{2}|AB| = \dfrac{1}{2}\sqrt{4^2+0^2+4^2} = \dfrac{1}{2}\sqrt{32} = 2\sqrt{2}$. (b) Height = distance from V to centre $= \sqrt{0^2+7^2+0^2} = 7$. Volume $= \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi(8)(7) = \dfrac{56\pi}{3}$.

6. (a) Segment area $= \dfrac{1}{2}r^2(\theta - \sin\theta) = \dfrac{1}{2}(9)(\theta-\sin\theta) = 4.5(\theta-\sin\theta)$. (b) At $\theta = 1.8$: $4.5(1.8 - \sin 1.8) \approx 3.72$ cm^2.

7. (a) Segment area $= \dfrac{1}{2}(2)^2(\theta-\sin\theta) = 2(\theta-\sin\theta)$. (b) Logo area = rectangle area $-2\times$ segment area: $35 - 4(\theta-\sin\theta) = 27.0 \Rightarrow \theta - \sin\theta = 2.0$. Solving numerically (GDC): $\theta \approx 2.55$ radians.

8. (a)(i) $AC^2 = AB^2+BC^2-2(AB)(BC)\cos\alpha = 4+36-24\cos\alpha = 40-24\cos\alpha$. (ii) $AC^2 = DA^2+CD^2-2(DA)(CD)\cos\beta = 100+64-160\cos\beta = 164-160\cos\beta$. (iii) Equating: $40-24\cos\alpha = 164-160\cos\beta \Rightarrow \cos\alpha = \dfrac{160\cos\beta-124}{24} = \dfrac{40\cos\beta-31}{6}$, so $\alpha = \arccos\left(\dfrac{40\cos\beta-31}{6}\right)$.

9. Area $= \dfrac{1}{2}(2)(6)\sin\alpha + \dfrac{1}{2}(8)(10)\sin\beta = 6\sin\alpha + 40\sin\beta$, subject to the relationship in Question 8(a)(iii). Maximising numerically (GDC) gives a maximum area of approximately 34.0 (square units), occurring at $\alpha \approx 2.31$, $\beta \approx 0.831$ radians.

10. (a) $A = \dfrac{1}{2}x(12-x)\sin\dfrac{2\pi}{3} = \dfrac{1}{2}x(12-x)\cdot\dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}}{4}x(12-x)$. (b) By the cosine rule, $y^2 = x^2 + (12-x)^2 - 2x(12-x)\cos\dfrac{2\pi}{3} = x^2+(12-x)^2+x(12-x)$ (since $\cos\frac{2\pi}{3} = -\frac{1}{2}$). Expanding: $y^2 = x^2+144-24x+x^2+12x-x^2 = x^2-12x+144$. (c) $A(x) = \dfrac{\sqrt{3}}{4}(12x-x^2)$; $A'(x) = \dfrac{\sqrt{3}}{4}(12-2x) = 0 \Rightarrow x = 6$. This is a maximum (the coefficient of x^2 in $12x-x^2$ is negative). $A(6) = \dfrac{\sqrt{3}}{4}(6)(6) = 9\sqrt{3}$ cm^2. (d) $y^2 = x^2-12x+144$ is an upward-opening parabola in x (coefficient of x^2 is $+1$), so its vertex at $x=6$ is a *minimum*, not a maximum: $y^2(6) = 36-72+144 = 108$. So y^2 is *not* maximised at $x = 6$ — in fact this is exactly where y^2 is smallest; on the open interval $0 < x < 12$, y^2 has no maximum (it approaches 144 as $x \to 0^+$ or $x \to 12^-$, but never attains it).