Geometry

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

(a) Convert $225°$ to radians, giving your answer as a multiple of π. **[1]**

(b) Convert $\dfrac{7\pi}{4}$ radians to degrees. **[2]**

2. [Maximum mark: 4] *Difficulty:* 1/5

A circle has radius 10 cm. Find

(a) the length of an arc that subtends an angle of 0.8 radians at the centre; **[2]**

(b) the area of the corresponding sector. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

A circle has centre O and radius 5 m. Points A and B lie on the circle such that $A\hat{O}B = 1.6$ radians.

(a) Find the length of the chord $[AB]$. **[3]**

(b) Find the area of the minor sector OAB. **[2]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Triangle ABC has $AB = 10$ cm, $BC = 13$ cm, and $A\hat{B}C = 55°$.

(a) Find AC. **[3]**

(b) Find the area of triangle ABC. **[3]**

5. [Maximum mark: 7] *Difficulty:* 3/5

Consider a circle with diameter AB, where A has coordinates $(1,0,0)$ and B has coordinates $(-1,0,6)$.

(a) **(i)** Find the coordinates of the centre of the circle. **[2]**

(ii) Find the radius of the circle. **[2]**

The circle forms the base of a right cone whose vertex V has coordinates $(0,6,3)$.

(b) Find the exact volume of the cone. **[3]**

6. [Maximum mark: 6] *Difficulty:* 3/5

A circle has centre O and radius 5 cm. Points C and D lie on the circle such that $C\hat{O}D = \theta$, where $0 < \theta < \pi$.

(a) Find an expression, in terms of θ, for the area of the minor segment cut off by chord $[CD]$. **[3]**

(b) Hence find the area of this segment when $\theta = 1.3$ radians. **[3]**

7. [Maximum mark: 8] *Difficulty:* 4/5

A company designs a logo by removing two equal segments from a rectangle measuring 8 cm by 6 cm. The points A and B lie on a circle, with centre O and radius 3 cm, such that $A\hat{O}B = \theta$, where $0 < \theta < \pi$.

(a) Find the area of one of the shaded segments in terms of θ. **[3]**

(b) Given that the area of the logo is 33.0 cm^2, find the value of θ. **[5]**

8. [Maximum mark: 7] *Difficulty:* 4/5

Consider a quadrilateral $ABCD$ such that $AB = 4$, $BC = 6$, $CD = 8$ and $DA = 10$. Let $\alpha = A\hat{B}C$ and $\beta = A\hat{D}C$.

(a) **(i)** Find AC^2 in terms of α. **[2]**

(ii) Find AC^2 in terms of β. **[2]**

(iii) Hence, or otherwise, find an expression for α in terms of β. **[3]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Using the quadrilateral $ABCD$ from Question 8, find the maximum area of $ABCD$. **[8]**

10. [Maximum mark: 13] *Difficulty:* 5/5

Challenge question. A right circular cone has a fixed slant height of 12 cm. Let the radius of the base be r cm and the height be h cm, where $0 < r < 12$.

(a) Show that $h^2 = 144 - r^2$. **[2]**

(b) Show that the volume of the cone is given by $V(r) = \dfrac{\pi}{3} r^2 \sqrt{144 - r^2}$. **[3]**

(c) Find the exact value of r that maximises $V(r)$. **[5]**

(d) Find the exact maximum volume, giving your answer in the form $k\sqrt{3}\pi$, where $k \in \mathbb{Z}^+$. **[3]**

Answer Key

1. (a) $225^\circ = \frac{5\pi}{4}$. (b) $\frac{7\pi}{4} = 315^\circ$.

2. (a) Arc length $= r\theta = 10(0.8) = 8$ cm. (b) Sector area $= \frac{1}{2}r^2\theta = \frac{1}{2}(100)(0.8) = 40$ cm^2.

3. (a) $AB^2 = 5^2+5^2-2(5)(5)\cos(1.6) = 50(1-\cos 1.6)$, so $AB \approx 7.17$ m. (b) Sector area $= \frac{1}{2}(25)(1.6) = 20$ m^2.

4. (a) $AC^2 = 10^2+13^2-2(10)(13)\cos 55^\circ \approx 119.9$, so $AC \approx 10.9$ cm. (b) Area $= \frac{1}{2}(10)(13)\sin 55^\circ \approx 53.2$ cm^2.

5. (a)(i) Centre $=$ midpoint of $AB = (0,0,3)$. (ii) Radius $= \frac{1}{2}|AB| = \frac{1}{2}\sqrt{2^2+0^2+6^2} = \frac{1}{2}\sqrt{40} = \sqrt{10}$. (b) Height $=$ distance from V to centre $= \sqrt{0^2+6^2+0^2} = 6$. Volume $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(10)(6) = 20\pi$.

6. (a) Segment area $= \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2}(25)(\theta-\sin\theta) = 12.5(\theta-\sin\theta)$. (b) At $\theta = 1.3$: $12.5(1.3 - \sin 1.3) \approx 4.21$ cm^2.

7. (a) Segment area $= \frac{1}{2}(3)^2(\theta-\sin\theta) = 4.5(\theta-\sin\theta)$. (b) Logo area $=$ rectangle area $-2\times$ segment area: $48 - 9(\theta-\sin\theta) = 33.0 \Rightarrow \theta - \sin\theta = 1.6\overline{6}$. Solving numerically (GDC): $\theta \approx 2.37$ radians.

8. (a)(i) $AC^2 = AB^2+BC^2-2(AB)(BC)\cos\alpha = 16+36-48\cos\alpha = 52-48\cos\alpha$. (ii) $AC^2 = DA^2+CD^2-2(DA)(CD)\cos\beta = 100+64-160\cos\beta = 164-160\cos\beta$. (iii) Equating: $52-48\cos\alpha = 164-160\cos\beta \Rightarrow \cos\alpha = \frac{160\cos\beta - 112}{48} = \frac{10\cos\beta - 7}{3}$, so $\alpha = \arccos\left(\frac{10\cos\beta-7}{3}\right)$.

9. Area $= \frac{1}{2}(4)(6)\sin\alpha + \frac{1}{2}(8)(10)\sin\beta = 12\sin\alpha + 40\sin\beta$, subject to the relationship in Question 8(a)(iii). Maximising numerically (GDC) gives a maximum area of approximately 43.8 (square units), occurring at $\alpha \approx 2.14$, $\beta \approx 1.00$ radians.

10. (a) By Pythagoras on the cone's cross-section, slant height^2 = r^2+h^2, so $144 = r^2+h^2 \Rightarrow h^2 = 144-r^2$. (b) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144-r^2}$. (c) Maximise $V^2 \propto r^4(144-r^2)$. Let $u = r^2$: $\frac{d}{du}\left[144u^2 - u^3\right] = 288u - 3u^2 = 0 \Rightarrow u(288-3u) = 0 \Rightarrow u = 96$ (rejecting $u = 0$). So $r^2 = 96 \Rightarrow r = \sqrt{96} = 4\sqrt{6}$. (d) $h^2 = 144-96 = 48 \Rightarrow h = 4\sqrt{3}$. $V = \frac{1}{3}\pi(96)(4\sqrt{3}) = 128\sqrt{3}\pi$.