Functions

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

---

1. [Maximum mark: 3] *Difficulty:* 1/5

State the domain of $f(x) = \sqrt{4x-8}$. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Let $f(x) = 2x+3$ and $g(x) = x^2 - 2$.

(a) Find $(f \circ g)(1)$. **[2]**

(b) Find $(g \circ f)(0)$. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Let $f(x) = 4x-3$.

(a) Find $f^{-1}(x)$. **[3]**

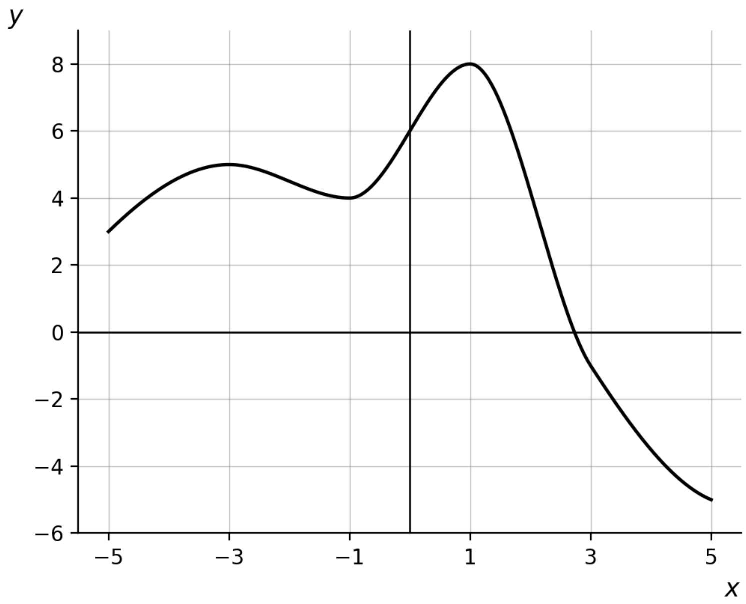
(b) Verify that $f\left(f^{-1}(x)\right) = x$. **[2]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Show that $f(x) = x^3 + 2x$ is an odd function. **[5]**

5. [Maximum mark: 6] *Difficulty:* 2/5

The graph of $y = f(x)$ for $-5 \leq x \leq 5$ is shown below.



(a) Write down the value of $f(-3)$. **[1]**

(b) Write down the value of $(f \circ f)(-3)$. **[2]**

(c) Let $g(x) = \frac{1}{5}f(x) + 3$. Find $g(-3)$. **[3]**

6. [Maximum mark: 9] *Difficulty:* 3/5

Let $f(x) = x^2 - 8x + 19$, where $x \geq 4$.

(a) Express $f(x)$ in the form $(x-p)^2 + q$, and hence state the range of f. **[3]**

(b) Find $f^{-1}(x)$, stating its domain and range. **[4]**

(c) Hence solve the equation $f^{-1}(x) = 9$. **[2]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Let $f(x) = |4x - 8|$.

(a) Solve the equation $f(x) = 16$. **[3]**

(b) Solve the inequality $f(x) < 8$. **[4]**

8. [Maximum mark: 7] *Difficulty:* 4/5

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = x^2 + 8x + k$, where $h, k \in \mathbb{R}$. The graphs of f and g have a common tangent line at $x = 5$.

(a) Find $f'(x)$. **[1]**

(b) Show that $h = 14$. **[3]**

(c) Hence show that $k = 146$. **[3]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Consider $f(x) = \sqrt{25 - x^2}$, for $0 \leq x \leq 5$.

(a) Find $f^{-1}(x)$, stating its domain. **[4]**

(b) Hence show that f is a self-inverse function. **[2]**

(c) State the geometric feature of the graph of f that explains this result. **[2]**

10. [Maximum mark: 16] *Difficulty: 5/5*

Challenge question. Consider the function $f(x) = \sqrt{x^2 - 16}$, where $4 \leq x \leq 5$.

(a) State the coordinates of the endpoints of the curve $y = f(x)$. **[2]**

(b) **(i)** Show that the inverse function is given by $f^{-1}(x) = \sqrt{x^2 + 16}$. **[5]**

(ii) State the domain and range of f^{-1}. **[2]**

The curve $y = f(x)$ is rotated 2π radians about the y-axis to form a solid of revolution used to model a water container.

(c) **(i)** Show that the volume, V m^3, of water in the container when filled to a height of h metres (where $0 \leq h \leq 3$) is given by

$$V = \pi\left(\frac{h^3}{3} + 16h\right).$$

[4]

(ii) Hence determine the maximum volume of the container. **[3]**

Answer Key

1. Require $4x - 8 \geq 0 \Rightarrow x \geq 2$.

2. (a) $g(1) = 1 - 2 = -1$, so $f(g(1)) = 2(-1) + 3 = 1$. (b) $f(0) = 2(0) + 3 = 3$, so $g(f(0)) = 3^2 - 2 = 7$.

3. (a) $y = 4x - 3 \Rightarrow x = \dfrac{y+3}{4}$, so $f^{-1}(x) = \dfrac{x+3}{4}$. (b) $f\left(f^{-1}(x)\right) = 4\left(\dfrac{x+3}{4}\right) - 3 = (x+3) - 3 = x$. ✓

4. $f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x = -(x^3 + 2x) = -f(x)$. Since $f(-x) = -f(x)$ for all x, f is odd. ■

5. (a) $f(-3) = 5$. (b) $(f \circ f)(-3) = f(5) = -5$. (c) $g(-3) = \dfrac{1}{5}(5) + 3 = 1 + 3 = 4$.

6. (a) $f(x) = (x-4)^2 + 3$; for $x \geq 4$, range is $f(x) \geq 3$. (b) $y = (x-4)^2 + 3 \Rightarrow x = 4 + \sqrt{y-3}$ (positive root, since $x \geq 4$), so $f^{-1}(x) = 4 + \sqrt{x-3}$, domain $x \geq 3$, range $y \geq 4$. (c) $4 + \sqrt{x-3} = 9 \Rightarrow \sqrt{x-3} = 5 \Rightarrow x - 3 = 25 \Rightarrow x = 28$.

7. (a) $4x - 8 = \pm 16 \Rightarrow x = 6$ or $x = -2$. (b) $-8 < 4x - 8 < 8 \Rightarrow 0 < 4x < 16 \Rightarrow 0 < x < 4$.

8. (a) $f'(x) = -2(x-h)$. (b) $g'(x) = 2x + 8$. At $x = 5$: $f'(5) = -2(5-h) = -10 + 2h$ and $g'(5) = 2(5) + 8 = 18$. Setting these equal: $-10 + 2h = 18 \Rightarrow h = 14$. (c) $f(5) = -(5-14)^2 + 2k = -81 + 2k$; $g(5) = 25 + 40 + k = 65 + k$. Setting these equal: $-81 + 2k = 65 + k \Rightarrow k = 146$.

9. (a) $y = \sqrt{25 - x^2} \Rightarrow x = \sqrt{25 - y^2}$ (positive root, since $0 \leq x \leq 5$). So $f^{-1}(x) = \sqrt{25 - x^2}$, with domain $0 \leq x \leq 5$. (b) $f^{-1}(x) = \sqrt{25 - x^2} = f(x)$ for all x in the domain, so f is self-inverse. (c) The graph of f (a quarter-circle arc of radius 5) is symmetric about the line $y = x$.

10. (a) At $x = 4$: $f(4) = 0$. At $x = 5$: $f(5) = 3$. Endpoints: $(4, 0)$ and $(5, 3)$. (b)(i) $y = \sqrt{x^2 - 16} \Rightarrow y^2 = x^2 - 16 \Rightarrow x^2 = y^2 + 16 \Rightarrow x = \sqrt{y^2 + 16}$ (positive root, since $x \geq 4$). Swapping x and y: $f^{-1}(x) = \sqrt{x^2 + 16}$. (ii) Domain of f^{-1} = range of f = $[0, 3]$; range of f^{-1} = domain of f = $[4, 5]$. (c)(i) Since $y = \sqrt{x^2 - 16}$, we have $x^2 = y^2 + 16$. Rotating about the y-axis, the volume up to height h is

$$V = \pi \int_0^h x^2 \, dy = \pi \int_0^h (y^2 + 16) \, dy = \pi \left[\frac{y^3}{3} + 16y\right]_0^h = \pi \left(\frac{h^3}{3} + 16h\right).$$

(ii) Maximum volume occurs at $h = 3$: $V = \pi \left(\dfrac{9}{3} + 48\right) = \pi(3 + 48) = 51\pi \approx 160 \text{ m}^3$.