Functions

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

State the domain of $f(x) = \sqrt{3x-9}$. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Let $f(x) = 4x-1$ and $g(x) = x^2+2$.

(a) Find $(f \circ g)(2)$. **[2]**

(b) Find $(g \circ f)(1)$. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Let $f(x) = 3x+4$.

(a) Find $f^{-1}(x)$. **[3]**

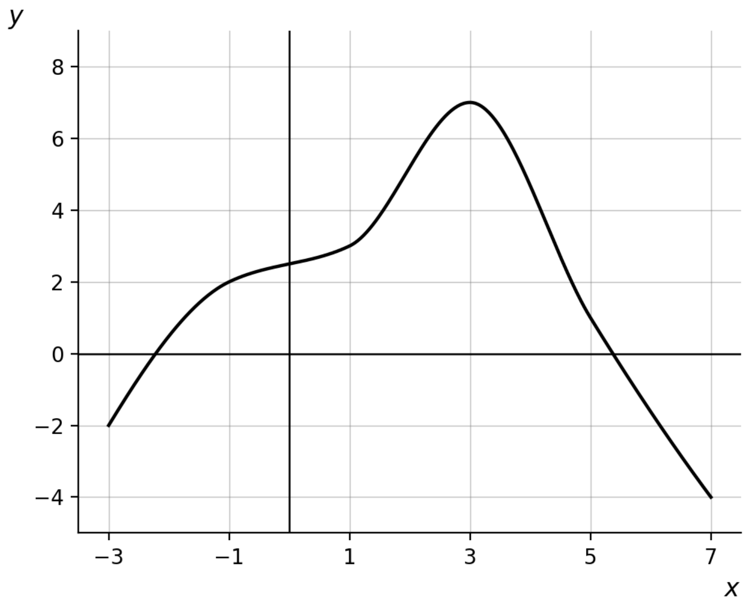
(b) Verify that $f\left(f^{-1}(x)\right) = x$. **[2]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Show that $f(x) = x^5 - 3x$ is an odd function. **[5]**

5. [Maximum mark: 6] *Difficulty:* 2/5

The graph of $y = f(x)$ for $-3 \leq x \leq 7$ is shown below.



(a) Write down the value of $f(3)$. **[1]**

(b) Write down the value of $(f \circ f)(3)$. **[2]**

(c) Let $g(x) = \frac{1}{7} f(x) + 2$. Find $g(3)$. **[3]**

6. [Maximum mark: 9] *Difficulty:* 3/5

Let $f(x) = x^2 - 6x + 13$, where $x \geq 3$.

(a) Express $f(x)$ in the form $(x-p)^2 + q$, and hence state the range of f. **[3]**

(b) Find $f^{-1}(x)$, stating its domain and range. **[4]**

(c) Hence solve the equation $f^{-1}(x) = 7$. **[2]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Let $f(x) = |3x - 9|$.

(a) Solve the equation $f(x) = 12$. **[3]**

(b) Solve the inequality $f(x) < 6$. **[4]**

8. [Maximum mark: 7] *Difficulty:* 4/5

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = x^2 + 6x + k$, where $h, k \in \mathbb{R}$. The graphs of f and g have a common tangent line at $x = 4$.

(a) Find $f'(x)$. **[1]**

(b) Show that $h = 11$. **[3]**

(c) Hence show that $k = 89$. **[3]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Consider $f(x) = \sqrt{16 - x^2}$, for $0 \leq x \leq 4$.

(a) Find $f^{-1}(x)$, stating its domain. **[4]**

(b) Hence show that f is a self-inverse function. **[2]**

(c) State the geometric feature of the graph of f that explains this result. **[2]**

10. [Maximum mark: 16] *Difficulty: 5/5*

Challenge question. Consider the function $f(x) = \sqrt{x^2 - 9}$, where $3 \leq x \leq 5$.

(a) State the coordinates of the endpoints of the curve $y = f(x)$. **[2]**

(b) **(i)** Show that the inverse function is given by $f^{-1}(x) = \sqrt{x^2 + 9}$. **[5]**

(ii) State the domain and range of f^{-1}. **[2]**

The curve $y = f(x)$ is rotated 2π radians about the y-axis to form a solid of revolution used to model a water container.

(c) **(i)** Show that the volume, V m^3, of water in the container when filled to a height of h metres (where $0 \leq h \leq 4$) is given by

$$V = \pi\left(\frac{h^3}{3} + 9h\right).$$

[4]

(ii) Hence determine the maximum volume of the container. **[3]**

Answer Key

1. Require $3x - 9 \geq 0 \Rightarrow x \geq 3$.

2. (a) $g(2) = 4 + 2 = 6$, so $f(g(2)) = 4(6) - 1 = 23$. (b) $f(1) = 4(1) - 1 = 3$, so $g(f(1)) = 3^2 + 2 = 11$.

3. (a) $y = 3x + 4 \Rightarrow x = \dfrac{y-4}{3}$, so $f^{-1}(x) = \dfrac{x-4}{3}$. (b) $f\left(f^{-1}(x)\right) = 3\left(\dfrac{x-4}{3}\right) + 4 = (x-4) + 4 = x$. ✓

4. $f(-x) = (-x)^5 - 3(-x) = -x^5 + 3x = -(x^5 - 3x) = -f(x)$. Since $f(-x) = -f(x)$ for all x, f is odd. ■

5. (a) $f(3) = 7$. (b) $(f \circ f)(3) = f(7) = -4$. (c) $g(3) = \dfrac{1}{7}(7) + 2 = 1 + 2 = 3$.

6. (a) $f(x) = (x-3)^2 + 4$; for $x \geq 3$, range is $f(x) \geq 4$. (b) $y = (x-3)^2 + 4 \Rightarrow x = 3 + \sqrt{y-4}$ (positive root, since $x \geq 3$), so $f^{-1}(x) = 3 + \sqrt{x-4}$, domain $x \geq 4$, range $y \geq 3$. (c) $3 + \sqrt{x-4} = 7 \Rightarrow \sqrt{x-4} = 4 \Rightarrow x - 4 = 16 \Rightarrow x = 20$.

7. (a) $3x - 9 = \pm 12 \Rightarrow x = 7$ or $x = -1$. (b) $-6 < 3x - 9 < 6 \Rightarrow 3 < 3x < 15 \Rightarrow 1 < x < 5$.

8. (a) $f'(x) = -2(x-h)$. (b) $g'(x) = 2x + 6$. At $x = 4$: $f'(4) = -2(4-h) = -8 + 2h$ and $g'(4) = 2(4) + 6 = 14$. Setting these equal: $-8 + 2h = 14 \Rightarrow h = 11$. (c) $f(4) = -(4-11)^2 + 2k = -49 + 2k$; $g(4) = 16 + 24 + k = 40 + k$. Setting these equal: $-49 + 2k = 40 + k \Rightarrow k = 89$.

9. (a) $y = \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2}$ (positive root, since $0 \leq x \leq 4$). So $f^{-1}(x) = \sqrt{16 - x^2}$, with domain $0 \leq x \leq 4$. (b) $f^{-1}(x) = \sqrt{16 - x^2} = f(x)$ for all x in the domain, so f is self-inverse. (c) The graph of f (a quarter-circle arc of radius 4) is symmetric about the line $y = x$.

10. (a) At $x = 3$: $f(3) = 0$. At $x = 5$: $f(5) = 4$. Endpoints: $(3,0)$ and $(5,4)$. (b)(i) $y = \sqrt{x^2 - 9} \Rightarrow y^2 = x^2 - 9 \Rightarrow x^2 = y^2 + 9 \Rightarrow x = \sqrt{y^2 + 9}$ (positive root, since $x \geq 3$). Swapping x and y: $f^{-1}(x) = \sqrt{x^2 + 9}$. (ii) Domain of f^{-1} = range of f = $[0,4]$; range of f^{-1} = domain of f = $[3,5]$. (c)(i) Since $y = \sqrt{x^2 - 9}$, we have $x^2 = y^2 + 9$. Rotating about the y-axis, the volume up to height h is

$$V = \pi \int_0^h x^2 \, dy = \pi \int_0^h (y^2 + 9) \, dy = \pi \left[\frac{y^3}{3} + 9y \right]_0^h = \pi \left(\frac{h^3}{3} + 9h \right).$$

(ii) Maximum volume occurs at $h = 4$: $V = \pi \left(\dfrac{64}{3} + 36 \right) = \pi \left(\dfrac{64 + 108}{3} \right) = \dfrac{172\pi}{3} \approx 180 \text{ m}^3$.