Functions

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

State the domain of $f(x) = \sqrt{2x-6}$. **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Let $f(x) = 3x+2$ and $g(x) = x^2 - 1$.

(a) Find $(f \circ g)(3)$. **[2]**

(b) Find $(g \circ f)(1)$. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Let $f(x) = 2x-5$.

(a) Find $f^{-1}(x)$. **[3]**

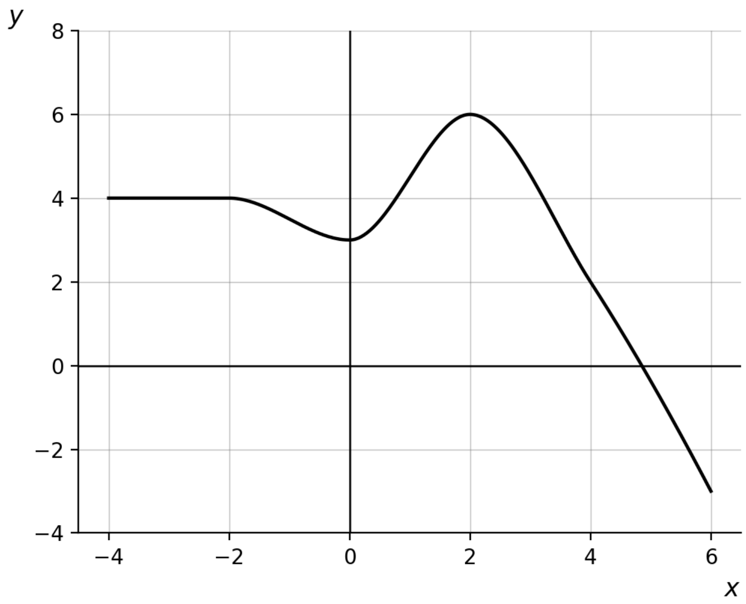
(b) Verify that $f\left(f^{-1}(x)\right) = x$. **[2]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Show that $f(x) = x^3 - 4x$ is an odd function. **[5]**

5. [Maximum mark: 6] *Difficulty:* 2/5

The graph of $y = f(x)$ for $-4 \leq x \leq 6$ is shown below.



(a) Write down the value of $f(2)$. **[1]**

(b) Write down the value of $(f \circ f)(2)$. **[2]**

(c) Let $g(x) = \frac{1}{6}f(x) + 1$. Find $g(2)$. **[3]**

6. [Maximum mark: 9] *Difficulty:* 3/5

Let $f(x) = x^2 - 4x + 7$, where $x \geq 2$.

(a) Express $f(x)$ in the form $(x-p)^2 + q$, and hence state the range of f. **[3]**

(b) Find $f^{-1}(x)$, stating its domain and range. **[4]**

(c) Hence solve the equation $f^{-1}(x) = 5$. **[2]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Let $f(x) = |2x - 6|$.

(a) Solve the equation $f(x) = 10$. **[3]**

(b) Solve the inequality $f(x) < 4$. **[4]**

8. [Maximum mark: 7] *Difficulty:* 4/5

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = x^2 + 4x + k$, where $h, k \in \mathbb{R}$. The graphs of f and g have a common tangent line at $x = 3$ (i.e. they intersect and have equal gradient there).

(a) Find $f'(x)$. **[1]**

(b) Show that $h = 8$. **[3]**

(c) Hence show that $k = 46$. **[3]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Consider $f(x) = \sqrt{9 - x^2}$, for $0 \leq x \leq 3$.

(a) Find $f^{-1}(x)$, stating its domain. **[4]**

(b) Hence show that f is a self-inverse function. **[2]**

(c) State the geometric feature of the graph of f that explains this result. **[2]**

10. [Maximum mark: 16] *Difficulty: 5/5*

Challenge question. Consider the function $f(x) = \sqrt{x^2 - 4}$, where $2 \leq x \leq 3$.

(a) State the coordinates of the endpoints of the curve $y = f(x)$. **[2]**

(b) **(i)** Show that the inverse function is given by $f^{-1}(x) = \sqrt{x^2 + 4}$. **[5]**

(ii) State the domain and range of f^{-1}. **[2]**

The curve $y = f(x)$ is rotated 2π radians about the y-axis to form a solid of revolution used to model a water container.

(c) **(i)** Show that the volume, V m^3, of water in the container when filled to a height of h metres (where $0 \leq h \leq \sqrt{5}$) is given by

$$V = \pi\left(\frac{h^3}{3} + 4h\right).$$

[4]

(ii) Hence determine the maximum volume of the container. **[3]**

Answer Key

1. Require $2x-6 \geq 0 \Rightarrow x \geq 3$.

2. (a) $g(3)=9-1=8$, so $f(g(3))=3(8)+2=26$. (b) $f(1)=3(1)+2=5$, so $g(f(1))=5^2-1=24$.

3. (a) $y=2x-5 \Rightarrow x=\dfrac{y+5}{2}$, so $f^{-1}(x)=\dfrac{x+5}{2}$. (b) $f\left(f^{-1}(x)\right)=2\left(\dfrac{x+5}{2}\right)-5=(x+5)-5=x$. ✓

4. $f(-x)=(-x)^3-4(-x)=-x^3+4x=-(x^3-4x)=-f(x)$. Since $f(-x)=-f(x)$ for all x, f is odd. ■

5. (a) $f(2)=6$. (b) $(f\circ f)(2)=f(6)=-3$. (c) $g(2)=\dfrac{1}{6}(6)+1=1+1=2$.

6. (a) $f(x)=(x-2)^2+3$; for $x\geq 2$, range is $f(x)\geq 3$. (b) $y=(x-2)^2+3 \Rightarrow x=2+\sqrt{y-3}$ (taking the positive root since $x\geq 2$), so $f^{-1}(x)=2+\sqrt{x-3}$, domain $x\geq 3$, range $y\geq 2$. (c) $2+\sqrt{x-3}=5 \Rightarrow \sqrt{x-3}=3 \Rightarrow x-3=9 \Rightarrow x=12$.

7. (a) $2x-6=\pm 10 \Rightarrow x=8$ or $x=-2$. (b) $-4<2x-6<4 \Rightarrow 2<2x<10 \Rightarrow 1<x<5$.

8. (a) $f'(x)=-2(x-h)$. (b) $g'(x)=2x+4$. At $x=3$: $f'(3)=-2(3-h)=-6+2h$ and $g'(3)=2(3)+4=10$. Setting these equal: $-6+2h=10 \Rightarrow h=8$. (c) $f(3)=-(3-8)^2+2k=-25+2k$; $g(3)=9+12+k=21+k$. Setting these equal (since the curves also meet at $x=3$): $-25+2k=21+k \Rightarrow k=46$.

9. (a) $y=\sqrt{9-x^2} \Rightarrow x=\sqrt{9-y^2}$ (positive root, since $0\leq x\leq 3$). So $f^{-1}(x)=\sqrt{9-x^2}$, with domain equal to the range of f, i.e. $0\leq x\leq 3$. (b) $f^{-1}(x)=\sqrt{9-x^2}=f(x)$ for all x in the domain, so f is self-inverse. (c) The graph of f (a quarter-circle arc of radius 3) is symmetric about the line $y=x$.

10. (a) At $x=2$: $f(2)=0$. At $x=3$: $f(3)=\sqrt{5}$. Endpoints: $(2,0)$ and $(3,\sqrt{5})$. (b)(i) $y=\sqrt{x^2-4} \Rightarrow y^2=x^2-4 \Rightarrow x^2=y^2+4 \Rightarrow x=\sqrt{y^2+4}$ (positive root, since $x\geq 2$). Swapping x and y: $f^{-1}(x)=\sqrt{x^2+4}$. (ii) Domain of f^{-1} = range of f = $[0,\sqrt{5}]$; range of f^{-1} = domain of f = $[2,3]$. (c)(i) Since $y=\sqrt{x^2-4}$, we have $x^2=y^2+4$. Rotating about the y-axis, the volume up to height h is

$$V=\pi\int_0^h x^2\,dy=\pi\int_0^h (y^2+4)\,dy=\pi\left[\frac{y^3}{3}+4y\right]_0^h=\pi\left(\frac{h^3}{3}+4h\right).$$

(ii) Maximum volume occurs at $h=\sqrt{5}$: $V=\pi\left(\dfrac{(\sqrt{5})^3}{3}+4\sqrt{5}\right)=\pi\left(\dfrac{5\sqrt{5}}{3}+4\sqrt{5}\right)=\dfrac{17\sqrt{5}\pi}{3}\approx 39.8$ m^3.