Differentiation

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

Let $f(x) = 2x^4 - 5x^2 + 3x - 1$.

(a) Find $f'(x)$. **[2]**

(b) Find $f'(1)$. **[2]**

2. [Maximum mark: 5] *Difficulty:* 1/5

Let $f(x) = (2x^3 + 1)^3$. Find $f'(x)$. **[5]**

3. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = x^4 e^{-x}$. Find $f'(x)$. **[6]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = \dfrac{5x+3}{2x^2+1}$. Find $f'(x)$. **[6]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = \sin^2(3x)$. Find $f'(x)$. **[6]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the curve $y = x^3 - 2x + 3$.

Find the equation of the tangent to the curve at the point where $x = -1$. **[7]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Consider the curve with equation $2x^2 + xy + y^2 = 8$.

(a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y. **[4]**

(b) Hence find the gradient of the curve at the point $(1, 2)$. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

The following diagram shows liquid in a container. Initially, the container is empty. Liquid is poured into the container at a rate of 5 cm^3s^{-1}.

The volume V cm^3 and height h cm of the liquid in the container satisfy the equation

$$V = 3\pi h^2 - \frac{1}{8}\pi h^3.$$

Find the rate of change of the height of the liquid at the instant when the volume of the liquid is 300 cm^3. **[8]**

9. [Maximum mark: 8] *Difficulty: 4/5*

Consider the function $f(x) = x^3 - 12x^2 + 36x + 4$.

(a) Find the coordinates of the stationary points of the graph of $y = f(x)$. **[4]**

(b) Use the second derivative to determine the nature of each stationary point. **[4]**

10. [Maximum mark: 13] *Difficulty: 5/5*

Challenge question. Consider the curve with equation $(x^2 + y^2)x^2 = 25y^2$, where $y \geq 0$ and $-5 < x < 5$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y. **[5]**

(b) Show that the curve has no local maximum or minimum points for $y > 0$. **[4]**

(c) Verify that the point $\left(2\sqrt{5}, 4\sqrt{5}\right)$ lies on the curve, and find the gradient of the curve at this point. **[4]**

Answer Key

1. (a) $f'(x) = 8x^3 - 10x + 3$. (b) $f'(1) = 8 - 10 + 3 = 1$.

2. $f'(x) = 3(2x^3+1)^2 \cdot 6x^2 = 18x^2(2x^3+1)^2$.

3. $f'(x) = 4x^3e^{-x} - x^4e^{-x} = x^3e^{-x}(4-x)$.

4. $f'(x) = \dfrac{5(2x^2+1) - (5x+3)(4x)}{(2x^2+1)^2} = \dfrac{10x^2+5-20x^2-12x}{(2x^2+1)^2} = \dfrac{-10x^2-12x+5}{(2x^2+1)^2}$.

5. $f'(x) = 2\sin(3x)\cos(3x) \cdot 3 = 6\sin(3x)\cos(3x)$ (equivalently $3\sin(6x)$).

6. At $x = -1$: $y = -1+2+3 = 4$. $y' = 3x^2 - 2$, so $y'(-1) = 3 - 2 = 1$. Tangent: $y - 4 = 1(x+1) \Rightarrow y = x + 5$.

7. (a) Differentiating implicitly: $4x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx}(x+2y) = -4x - y \Rightarrow \dfrac{dy}{dx} = -\dfrac{4x+y}{x+2y}$. (b) At $(1,2)$: $\dfrac{dy}{dx} = -\dfrac{4(1)+2}{1+2(2)} = -\dfrac{6}{5}$.

8. $\dfrac{dV}{dh} = 6\pi h - \dfrac{3}{8}\pi h^2$. Solving $3\pi h^2 - \dfrac{1}{8}\pi h^3 = 300$ numerically (GDC) gives $h \approx 6.632$. At this h: $\dfrac{dV}{dh} \approx 23.30\pi \approx 73.2$. Since $\dfrac{dh}{dt} = \dfrac{dV/dt}{dV/dh}$: $\dfrac{dh}{dt} \approx \dfrac{5}{73.2} \approx 0.0683$ cm s^{-1}.

9. (a) $f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6)$, so stationary points at $x = 2$ and $x = 6$. $f(2) = 8 - 48 + 72 + 4 = 36$; $f(6) = 216 - 432 + 216 + 4 = 4$. Stationary points: $(2,36)$ and $(6,4)$. (b) $f''(x) = 6x - 24$. $f''(2) = -12 < 0$, so $(2,36)$ is a local maximum. $f''(6) = 12 > 0$, so $(6,4)$ is a local minimum.

10. (a) Differentiating implicitly: $4x^3 + 2xy^2 + 2x^2y\dfrac{dy}{dx} = 50y\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx}(2x^2y - 50y) = -4x^3 - 2xy^2 \Rightarrow \dfrac{dy}{dx} = \dfrac{x(2x^2+y^2)}{y(25-x^2)}$. (b) For $\dfrac{dy}{dx} = 0$ (with $y \neq 0$ and $x \neq \pm 5$), we need $x(2x^2+y^2) = 0$. Since $2x^2 + y^2 > 0$ unless $x = y = 0$, this requires $x = 0$. Substituting $x = 0$ into the original equation gives $y^2 \cdot 0 = 25y^2 \Rightarrow y = 0$, which contradicts $y > 0$. So there is no point on the curve with $y > 0$ where $\dfrac{dy}{dx} = 0$; hence the curve has no local maximum or minimum points for $y > 0$. ■ (c) At $\left(2\sqrt{5}, 4\sqrt{5}\right)$: $x^2 = 20$, $y^2 = 80$, so $(x^2+y^2)x^2 = (100)(20) = 2000$ and $25y^2 = 25(80) = 2000$, confirming the point lies on the curve. Gradient: $\dfrac{dy}{dx} = \dfrac{2\sqrt{5}(2(20)+80)}{4\sqrt{5}(25-20)} = \dfrac{2\sqrt{5}(120)}{4\sqrt{5}(5)} = 12$.