Differentiation

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

Let $f(x) = 4x^3 + 2x^2 - 6x + 5$.

(a) Find $f'(x)$. **[2]**

(b) Find $f'(-1)$. **[2]**

2. [Maximum mark: 5] *Difficulty:* 1/5

Let $f(x) = (4x^2 - 1)^4$. Find $f'(x)$. **[5]**

3. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = x^2 e^{3x}$. Find $f'(x)$. **[6]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = \dfrac{3x-2}{x^2+1}$. Find $f'(x)$. **[6]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = \ln(\cos(2x))$. Find $f'(x)$. **[6]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the curve $y = x^3 - 3x + 2$.

Find the equation of the tangent to the curve at the point where $x = 2$. **[7]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Consider the curve with equation $x^2 + 2xy + 3y^2 = 6$.

(a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y. **[4]**

(b) Hence find the gradient of the curve at the point $(1, 1)$. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

The following diagram shows liquid in a container. Initially, the container is empty. Liquid is poured into the container at a rate of 4 cm^3s^{-1}.

The volume V cm^3 and height h cm of the liquid in the container satisfy the equation

$$V = 6\pi h^2 - \frac{1}{4}\pi h^3.$$

Find the rate of change of the height of the liquid at the instant when the volume of the liquid is 250 cm^3. **[8]**

9. [Maximum mark: 8] *Difficulty: 4/5*

Consider the function $f(x) = x^3 - 9x^2 + 24x - 5$.

(a) Find the coordinates of the stationary points of the graph of $y = f(x)$. **[4]**

(b) Use the second derivative to determine the nature of each stationary point. **[4]**

10. [Maximum mark: 13] *Difficulty: 5/5*

Challenge question. Consider the curve with equation $(x^2 + y^2)x^2 = 16y^2$, where $y \geq 0$ and $-4 < x < 4$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y. **[5]**

(b) Show that the curve has no local maximum or minimum points for $y > 0$. **[4]**

(c) Verify that the point $\left(2\sqrt{2}, 2\sqrt{2}\right)$ lies on the curve, and find the gradient of the curve at this point. **[4]**

Answer Key

1. (a) $f'(x) = 12x^2 + 4x - 6$. (b) $f'(-1) = 12 - 4 - 6 = 2$.

2. $f'(x) = 4(4x^2-1)^3 \cdot 8x = 32x(4x^2-1)^3$.

3. $f'(x) = 2xe^{3x} + x^2 \cdot 3e^{3x} = xe^{3x}(2+3x)$.

4. $f'(x) = \dfrac{3(x^2+1)-(3x-2)(2x)}{(x^2+1)^2} = \dfrac{3x^2+3-6x^2+4x}{(x^2+1)^2} = \dfrac{-3x^2+4x+3}{(x^2+1)^2}$.

5. $f'(x) = \dfrac{-\sin(2x)\cdot 2}{\cos(2x)} = -2\tan(2x)$.

6. At $x=2$: $y = 8-6+2 = 4$. $y' = 3x^2-3$, so $y'(2) = 12-3 = 9$. Tangent: $y-4 = 9(x-2) \Rightarrow y = 9x-14$.

7. (a) Differentiating implicitly: $2x+2y+2x\dfrac{dy}{dx}+2y+6y\dfrac{dy}{dx} = 0$; more carefully, differentiating $x^2+2xy+3y^2 = 6$: $2x+2\left(y+x\dfrac{dy}{dx}\right)+6y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx}(2x+6y) = -2x-2y \Rightarrow \dfrac{dy}{dx} = -\dfrac{x+y}{x+3y}$. (b) At $(1,1)$: $\dfrac{dy}{dx} = -\dfrac{1+1}{1+3} = -\dfrac{1}{2}$.

8. $\dfrac{dV}{dh} = 12\pi h - \dfrac{3}{4}\pi h^2$. Solving $6\pi h^2 - \dfrac{1}{4}\pi h^3 = 250$ numerically (GDC) gives $h \approx 3.988$. At this h: $\dfrac{dV}{dh} \approx 35.93\pi \approx 112.9$. Since $\dfrac{dh}{dt} = \dfrac{dV/dt}{dV/dh}$: $\dfrac{dh}{dt} \approx \dfrac{4}{112.9} \approx 0.0354$ cm s^{-1}.

9. (a) $f'(x) = 3x^2-18x+24 = 3(x-2)(x-4)$, so stationary points at $x=2$ and $x=4$. $f(2) = 8-36+48-5 = 15$; $f(4) = 64-144+96-5 = 11$. Stationary points: $(2,15)$ and $(4,11)$. (b) $f''(x) = 6x-18$. $f''(2) = -6 < 0$, so $(2,15)$ is a local maximum. $f''(4) = 6 > 0$, so $(4,11)$ is a local minimum.

10. (a) Differentiating implicitly: $4x^3+2xy^2+2x^2y\dfrac{dy}{dx} = 32y\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx}(2x^2y-32y) = -4x^3-2xy^2 \Rightarrow \dfrac{dy}{dx} = \dfrac{x(2x^2+y^2)}{y(16-x^2)}$. (b) For $\dfrac{dy}{dx} = 0$ (with $y \neq 0$ and $x \neq \pm 4$), we need $x(2x^2+y^2) = 0$. Since $2x^2+y^2 > 0$ unless $x = y = 0$, this requires $x = 0$. Substituting $x = 0$ into the original equation gives $y^2 \cdot 0 = 16y^2 \Rightarrow y = 0$, which contradicts $y > 0$. So there is no point on the curve with $y > 0$ where $\dfrac{dy}{dx} = 0$; hence the curve has no local maximum or minimum points for $y > 0$. ■ (c) At $\left(2\sqrt{2}, 2\sqrt{2}\right)$: $x^2 = y^2 = 8$, so $(x^2+y^2)x^2 = (16)(8) = 128$ and $16y^2 = 16(8) = 128$, confirming the point lies on the curve. Gradient: $\dfrac{dy}{dx} = \dfrac{2\sqrt{2}(2(8)+8)}{2\sqrt{2}(16-8)} = \dfrac{2\sqrt{2}(24)}{2\sqrt{2}(8)} = 3$.