Differentiation

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

Let $f(x) = 3x^4 - 5x^2 + 7x - 2$.

(a) Find $f'(x)$. **[2]**

(b) Find $f'(1)$. **[2]**

2. [Maximum mark: 5] *Difficulty:* 1/5

Let $f(x) = (3x^2 - 4)^5$. Find $f'(x)$. **[5]**

3. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = x^3 e^{2x}$. Find $f'(x)$. **[6]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = \dfrac{2x+1}{x^2+3}$. Find $f'(x)$. **[6]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Let $f(x) = e^{\sin(2x)}$. Find $f'(x)$. **[6]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the curve $y = x^3 - 4x + 1$.

Find the equation of the tangent to the curve at the point where $x = 2$. **[7]**

7. [Maximum mark: 7] *Difficulty:* 3/5

Consider the curve with equation $x^2 + xy + y^2 = 7$.

(a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y. **[4]**

(b) Hence find the gradient of the curve at the point $(1,2)$. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

The following diagram shows liquid in a container. Initially, the container is empty. Liquid is poured into the container at a rate of 3 cm^3s^{-1}.

The volume V cm^3 and height h cm of the liquid in the container satisfy the equation

$$V = 4\pi h^2 - \frac{1}{6}\pi h^3.$$

Find the rate of change of the height of the liquid at the instant when the volume of the liquid is 150 cm^3. **[8]**

9. [Maximum mark: 8] *Difficulty: 4/5*

Consider the function $f(x) = x^3 - 6x^2 + 9x + 2$.

(a) Find the coordinates of the stationary points of the graph of $y = f(x)$. **[4]**

(b) Use the second derivative to determine the nature of each stationary point. **[4]**

10. [Maximum mark: 13] *Difficulty: 5/5*

Challenge question. Consider the curve with equation $(x^2 + y^2)x^2 = 9y^2$, where $y \geq 0$ and $-3 < x < 3$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y. **[5]**

(b) Show that the curve has no local maximum or minimum points for $y > 0$. **[4]**

(c) Verify that the point $\left(\sqrt{5}, \frac{5}{2}\right)$ lies on the curve, and find the gradient of the curve at this point. **[4]**

Answer Key

1. (a) $f'(x) = 12x^3 - 10x + 7$. (b) $f'(1) = 12 - 10 + 7 = 9$.

2. $f'(x) = 5(3x^2 - 4)^4 \cdot 6x = 30x(3x^2 - 4)^4$.

3. $f'(x) = 3x^2 e^{2x} + x^3 \cdot 2e^{2x} = x^2 e^{2x}(3 + 2x)$.

4. $f'(x) = \dfrac{2(x^2+3) - (2x+1)(2x)}{(x^2+3)^2} = \dfrac{2x^2 + 6 - 4x^2 - 2x}{(x^2+3)^2} = \dfrac{-2x^2 - 2x + 6}{(x^2+3)^2} = \dfrac{-2(x^2 + x - 3)}{(x^2+3)^2}$.

5. $f'(x) = e^{\sin(2x)} \cdot \cos(2x) \cdot 2 = 2\cos(2x)e^{\sin(2x)}$.

6. At $x = 2$: $y = 8 - 8 + 1 = 1$. $y' = 3x^2 - 4$, so $y'(2) = 12 - 4 = 8$. Tangent: $y - 1 = 8(x-2) \Rightarrow y = 8x - 15$.

7. (a) Differentiating implicitly: $2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx}(x + 2y) = -2x - y \Rightarrow \dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$. (b) At $(1,2)$: $\dfrac{dy}{dx} = -\dfrac{2(1)+2}{1+2(2)} = -\dfrac{4}{5}$.

8. $\dfrac{dV}{dh} = 8\pi h - \dfrac{1}{2}\pi h^2$. Solving $4\pi h^2 - \dfrac{1}{6}\pi h^3 = 150$ numerically (GDC) gives $h \approx 3.762$. At this h: $\dfrac{dV}{dh} \approx 35.93\pi \approx 112.9$. Since $\dfrac{dh}{dt} = \dfrac{dV/dt}{dV/dh}$: $\dfrac{dh}{dt} \approx \dfrac{3}{112.9} \approx 0.0266$ cm s^{-1}.

9. (a) $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, so stationary points at $x = 1$ and $x = 3$. $f(1) = 1 - 6 + 9 + 2 = 6$; $f(3) = 27 - 54 + 27 + 2 = 2$. Stationary points: $(1,6)$ and $(3,2)$. (b) $f''(x) = 6x - 12$. $f''(1) = -6 < 0$, so $(1,6)$ is a local maximum. $f''(3) = 6 > 0$, so $(3,2)$ is a local minimum.

10. (a) Differentiating implicitly: $4x^3 + 2xy^2 + 2x^2y\dfrac{dy}{dx} = 18y\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx}(2x^2y - 18y) = -4x^3 - 2xy^2 \Rightarrow \dfrac{dy}{dx} = \dfrac{4x^3 + 2xy^2}{18y - 2x^2y} = \dfrac{x(2x^2 + y^2)}{y(9 - x^2)}$. (b) For $\dfrac{dy}{dx} = 0$ (with $y \neq 0$ and $x \neq \pm 3$), we need the numerator $x(2x^2 + y^2) = 0$. Since $2x^2 + y^2 > 0$ unless $x = y = 0$, this requires $x = 0$. But substituting $x = 0$ into the original equation gives $(0 + y^2)(0) = 9y^2 \Rightarrow 0 = 9y^2 \Rightarrow y = 0$, which contradicts $y > 0$. So there is no point on the curve with $y > 0$ where $\dfrac{dy}{dx} = 0$; hence the curve has no local maximum or minimum points for $y > 0$. ■ (c) At $\left(\sqrt{5}, \frac{5}{2}\right)$: $(x^2 + y^2)x^2 = (5 + 6.25)(5) = 56.25$ and $9y^2 = 9(6.25) = 56.25$, confirming the point lies on the curve. Gradient: $\dfrac{dy}{dx} = \dfrac{\sqrt{5}(2(5) + 6.25)}{\frac{5}{2}(9-5)} = \dfrac{\sqrt{5}(16.25)}{10} = \dfrac{13\sqrt{5}}{8}$.