Differential Equations

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 5] *Difficulty:* 1/5

Solve the differential equation $\dfrac{dy}{dx} = 5x^4y$, given that $y = 1$ when $x = 0$. **[5]**

2. [Maximum mark: 6] *Difficulty:* 1/5

The population P of insects in a greenhouse satisfies $\dfrac{dP}{dt} = 0.06P$, where t is measured in weeks. Initially, $P = 250$.

Find the population after 12 weeks, correct to the nearest whole number. **[6]**

3. [Maximum mark: 7] *Difficulty:* 2/5

Solve the differential equation $\dfrac{dy}{dx} + \dfrac{4}{x}y = x^2$, where $x > 0$, given that $y = 3$ when $x = 1$. Give your answer in the form $y = f(x)$. **[7]**

4. [Maximum mark: 8] *Difficulty:* 2/5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{2x^2 - y^2}{xy}$, where $x, y \neq 0$. It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y = f(x)$. **[6]**

(b) The points of zero gradient on the curve $y = f(x)$ lie on two straight lines of the form $y = mx$, where $m \in \mathbb{R}$. Find the values of m. **[2]**

5. [Maximum mark: 6] *Difficulty:* 3/5

Consider the differential equation $\dfrac{dy}{dx} = 2x - y$, with $y = 1$ when $x = 0$.

Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 0.3$. **[6]**

6. [Maximum mark: 9] *Difficulty:* 3/5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{3y}{x} - x^2$, where $x > 0$. It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method, with a step length of 0.2, to find an approximate value of y when $x = 2$. **[4]**

(b) By using the substitution $y = vx$, show that the exact solution is $y = x^3(2 - \ln x)$. **[4]**

(c) Find the actual value of y when $x = 2$, correct to three significant figures, and compare this with your answer to part (a). **[1]**

7. [Maximum mark: 7] *Difficulty:* 4/5

A bowl of soup cools according to Newton's law of cooling: $\dfrac{dT}{dt} = -k(T - 10)$, where T is the temperature of the soup in °C, t is time in minutes, and the room temperature is 10°C.

Initially the soup has temperature 95°C, and after 6 minutes its temperature is 55°C.

(a) Find the value of k. **[4]**

(b) Find the temperature of the soup after a further 6 minutes (i.e. at $t = 12$). **[3]**

8. [Maximum mark: 7] *Difficulty:* 4/5

A radioactive substance decays according to $\dfrac{dm}{dt} = -\lambda m$, where m is the mass remaining and t is time in years. The substance has a half-life of 5 years.

(a) Find the value of λ. **[2]**

(b) Find the time taken for the mass to decay to 20% of its original value. **[5]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Solve the differential equation $(x^2 + 1)\dfrac{dy}{dx} = xy$, given that $y = 6$ when $x = 0$. Give your answer in the form $y = f(x)$. **[8]**

10. [Maximum mark: 21] *Difficulty:* 5/5

Challenge question. The population, P, of insects in a greenhouse can be modelled by the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

where t is the time measured in weeks and k, N are positive constants. The constant N represents the maximum population that the greenhouse can sustain indefinitely.

(a) In the context of the population model, interpret the meaning of $\dfrac{dP}{dt}$. **[1]**

(b) Show that $\dfrac{d^2P}{dt^2} = k^2P\left(1 - \dfrac{P}{N}\right)\left(1 - \dfrac{2P}{N}\right)$. **[4]**

(c) Hence show that the population increases at its maximum rate when $P = \dfrac{N}{2}$. Justify your answer. **[5]**

(d) Hence determine the maximum value of $\dfrac{dP}{dt}$ in terms of k and N. **[2]**

Let P_0 be the initial population of insects.

(e) By solving the logistic differential equation, show that its solution can be expressed in the form

$$kt = \ln\left[\frac{P}{P_0} \cdot \frac{N - P_0}{N - P}\right].$$

[7]

After 5 weeks, the population of insects is $2P_0$. It is known that $N = 8P_0$.

(f) Find the value of k for this population model, correct to three significant figures. **[2]**

Answer Key

1. $\int \frac{dy}{y} = \int 5x^4\,dx \Rightarrow \ln y = x^5 + C$. At $(0,1)$: $\ln 1 = C = 0$. So $y = e^{x^5}$.

2. $P = 250e^{0.06t}$. $P(12) = 250e^{0.72} \approx 514$.

3. Integrating factor: $\mu = e^{\int(4/x)dx} = x^4$. $\frac{d}{dx}(x^4y) = x^4 \cdot x^2 = x^6$. $x^4y = \frac{x^7}{7} + C$. At $(1,3)$: $3 = \frac{1}{7} + C \Rightarrow C = \frac{20}{7}$. So $y = \frac{x^7/7 + 20/7}{x^4} = \frac{x^7+20}{7x^4}$.

4. (a) Let $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$. RHS $= \frac{2x^2 - v^2x^2}{x \cdot vx} = \frac{2-v^2}{v}$. So $x\frac{dv}{dx} = \frac{2-v^2}{v} - v = \frac{2-2v^2}{v}$. Separating: $\int \frac{v}{2-2v^2}dv = \int \frac{dx}{x} \Rightarrow -\frac{1}{4}\ln|2-2v^2| = \ln|x| + C \Rightarrow 2 - 2v^2 = \frac{A}{x^4}$. Since $v = y/x$: $2 - \frac{2y^2}{x^2} = \frac{A}{x^4} \Rightarrow y^2 = x^2 - \frac{A}{2x^2}$. At $(1,2)$: $4 = 1 - \frac{A}{2} \Rightarrow A = -6$. So $y^2 = x^2 + \frac{3}{x^2} = \frac{x^4+3}{x^2}$, giving $y = \frac{\sqrt{x^4+3}}{x}$. (b) Zero gradient requires the numerator $2 - v^2 = 0 \Rightarrow v = \pm\sqrt{2}$. So $m = \pm\sqrt{2}$.

5. Step 1: $y_1 = 1 + 0.1(0-1) = 0.9$. Step 2: $y_2 = 0.9 + 0.1(0.2 - 0.9) = 0.83$. Step 3: $y_3 = 0.83 + 0.1(0.4 - 0.83) = 0.787$. So $y(0.3) \approx 0.787$.

6. (a) Step 1 ($x = 1.2$): $y_1 = 2 + 0.2(3(2)-1) = 3.0$. Step 2 ($x = 1.4$): $y_2 = 3.0 + 0.2\left(\frac{3(3.0)}{1.2} - 1.2^2\right) \approx 4.212$. Step 3 ($x = 1.6$): $y_3 \approx 4.212 + 0.2\left(\frac{3(4.212)}{1.4} - 1.4^2\right) \approx 5.625$. Step 4 ($x = 1.8$): $y_4 \approx 5.625 + 0.2\left(\frac{3(5.625)}{1.6} - 1.6^2\right) \approx 7.223$. Step 5 ($x = 2.0$): $y_5 \approx 7.223 + 0.2\left(\frac{3(7.223)}{1.8} - 1.8^2\right) \approx 8.982$. So the Euler approximation gives $y(2) \approx 8.98$. (b) Let $y = vx$: $v + x\frac{dv}{dx} = 3v - x^2 \Rightarrow x\frac{dv}{dx} = 2v - x^2 \Rightarrow \frac{dv}{dx} - \frac{2v}{x} = -x$. This is linear in v, with integrating factor $\mu = e^{-\int(2/x)dx} = x^{-2}$. So $\frac{d}{dx}\left(\frac{v}{x^2}\right) = \frac{-x}{x^2} = -\frac{1}{x} \Rightarrow \frac{v}{x^2} = -\ln x + C \Rightarrow v = x^2(C - \ln x)$. Since $v = y/x$: $y = x^3(C - \ln x)$. At $(1,2)$: $2 = C \Rightarrow C = 2$. So $y = x^3(2 - \ln x)$. (c) $y(2) = 8(2 - \ln 2) \approx 10.5$. The Euler approximation (≈ 8.98) underestimates the actual value.

7. (a) $T - 10 = 85e^{-kt}$. At $t = 6$, $T = 55$: $45 = 85e^{-6k} \Rightarrow k = \frac{1}{6}\ln\frac{85}{45} = \frac{1}{6}\ln\frac{17}{9} \approx 0.106$. (b) $T(12) = 10 + 85e^{-12k} = 10 + 85\left(\frac{9}{17}\right)^2 = 10 + 85\left(\frac{81}{289}\right) = \frac{575}{17} \approx 33.8$°C.

8. (a) $\lambda = \frac{\ln 2}{5} \approx 0.139$. (b) $0.2 = e^{-\lambda t} \Rightarrow t = \frac{\ln 5}{\lambda} = \frac{5\ln 5}{\ln 2} \approx 11.6$ years.

9. $\int \frac{dy}{y} = \int \frac{x}{x^2+1}dx \Rightarrow \ln y = \frac{1}{2}\ln(x^2+1) + C \Rightarrow y = A\sqrt{x^2+1}$. At $(0,6)$: $6 = A$. So $y = 6\sqrt{x^2+1}$.

10. (a) $\frac{dP}{dt}$ represents the rate of change (growth rate) of the insect population with respect to time. (b) Differentiating $\frac{dP}{dt} = kP - \frac{kP^2}{N}$ with respect to t: $\frac{d^2P}{dt^2} = \left(k - \frac{2kP}{N}\right)\frac{dP}{dt} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$. (c) The growth rate is maximised where $\frac{d^2P}{dt^2} = 0$. This occurs at $P = 0$, $P = N$, or $P = \frac{N}{2}$. Since $P = 0$ and $P = N$ give $\frac{dP}{dt} = 0$ (the minimum possible growth rate), the maximum must occur at $P = \frac{N}{2}$, confirmed by checking the sign change of $\frac{d^2P}{dt^2}$ either side of this value. (d) At $P = \frac{N}{2}$: $\frac{dP}{dt} = k\left(\frac{N}{2}\right)\left(\frac{1}{2}\right) = \frac{kN}{4}$. (e) Separating variables and using partial fractions $\frac{1}{P(1-P/N)} = \frac{1}{P} + \frac{1}{N-P}$: $\ln P - \ln(N-P) = kt + C$.

At $t = 0$, $P = P_0$: $C = \ln P_0 - \ln(N - P_0)$. So $kt = \ln\left[\frac{P}{P_0} \cdot \frac{N - P_0}{N - P}\right]$. (f) With $P = 2P_0$, $N = 8P_0$, $t = 5$: $5k = \ln\left[2 \cdot \frac{7P_0}{6P_0}\right] = \ln\frac{7}{3}$. So $k = \frac{1}{5}\ln\frac{7}{3} \approx 0.169$.