Differential Equations

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 5] *Difficulty:* 1/5

Solve the differential equation $\frac{dy}{dx} = 3x^2y$, given that $y = 2$ when $x = 0$. **[5]**

2. [Maximum mark: 6] *Difficulty:* 1/5

The population P of a colony of bacteria satisfies $\frac{dP}{dt} = 0.05P$, where t is measured in hours. Initially, $P = 200$.

Find the population after 10 hours, correct to the nearest whole number. **[6]**

3. [Maximum mark: 7] *Difficulty:* 2/5

Solve the differential equation $\frac{dy}{dx} + \frac{2}{x}y = x^2$, where $x > 0$, given that $y = 1$ when $x = 1$. Give your answer in the form $y = f(x)$. **[7]**

4. [Maximum mark: 8] *Difficulty:* 2/5

Consider the differential equation $\frac{dy}{dx} = \frac{2y^2 - x^2}{xy}$, where $x, y \neq 0$. It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. **[6]**

(b) The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$, where $m \in \mathbb{R}$. Find the values of m. **[2]**

5. [Maximum mark: 6] *Difficulty:* 3/5

Consider the differential equation $\frac{dy}{dx} = x + y$, with $y = 1$ when $x = 0$.

Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 0.3$. **[6]**

6. [Maximum mark: 9] *Difficulty:* 3/5

Consider the differential equation $\frac{dy}{dx} = 1 + \frac{y}{x}$, where $x > 0$. It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method, with a step length of 0.25, to find an approximate value of y when $x = 2$. **[4]**

(b) By using the substitution $y = vx$, show that the exact solution is $y = x(\ln x + 2)$. **[4]**

(c) Find the actual value of y when $x = 2$, correct to three significant figures, and compare this with your answer to part (a). **[1]**

7. [Maximum mark: 7] *Difficulty:* 4/5

A cup of tea cools according to Newton's law of cooling: $\frac{dT}{dt} = -k(T - 20)$, where T is the temperature of the tea in °C, t is time in minutes, and the room temperature is 20°C.

Initially the tea has temperature 100°C, and after 5 minutes its temperature is 60°C.

(a) Find the value of k. **[4]**

(b) Find the temperature of the tea after a further 5 minutes (i.e. at $t = 10$). **[3]**

8. [Maximum mark: 7] *Difficulty:* 4/5

A radioactive substance decays according to $\frac{dm}{dt} = -\lambda m$, where m is the mass remaining and t is time in years. The substance has a half-life of 6 years.

(a) Find the value of λ. **[2]**

(b) Find the time taken for the mass to decay to 10% of its original value. **[5]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Solve the differential equation $(x^2 + 1)\frac{dy}{dx} = xy$, given that $y = 5$ when $x = 0$. Give your answer in the form $y = f(x)$. **[8]**

10. [Maximum mark: 21] *Difficulty:* 5/5

Challenge question. The population, P, of a colony of bacteria in a controlled culture can be modelled by the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

where t is the time measured in hours and k, N are positive constants. The constant N represents the maximum population that the culture can sustain indefinitely.

(a) In the context of the population model, interpret the meaning of $\frac{dP}{dt}$. **[1]**

(b) Show that $\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$. **[4]**

(c) Hence show that the population increases at its maximum rate when $P = \frac{N}{2}$. Justify your answer. **[5]**

(d) Hence determine the maximum value of $\frac{dP}{dt}$ in terms of k and N. **[2]**

Let P_0 be the initial population of bacteria.

(e) By solving the logistic differential equation, show that its solution can be expressed in the form

$$kt = \ln\left[\frac{P}{P_0} \cdot \frac{N - P_0}{N - P}\right].$$

[7]

After 8 hours, the population of bacteria is $2.5P_0$. It is known that $N = 5P_0$.

(f) Find the value of k for this population model. **[2]**

Answer Key

1. $\int \frac{dy}{y} = \int 3x^2\,dx \Rightarrow \ln y = x^3 + C$. At $(0,2)$: $\ln 2 = C$. So $y = 2e^{x^3}$.

2. $P = 200e^{0.05t}$. $P(10) = 200e^{0.5} \approx 330$.

3. Integrating factor: $\mu = e^{\int (2/x)dx} = x^2$. $\frac{d}{dx}(x^2y) = x^2 \cdot x^2 = x^4$. $x^2y = \frac{x^5}{5} + C$. At $(1,1)$: $1 = \frac{1}{5} + C \Rightarrow C = \frac{4}{5}$. So $y = \frac{x^5/5 + 4/5}{x^2} = \frac{x^5+4}{5x^2}$.

4. (a) Let $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$. RHS $= \frac{2v^2x^2 - x^2}{x \cdot vx} = \frac{2v^2-1}{v}$. So $x\frac{dv}{dx} = \frac{2v^2-1}{v} - v = \frac{v^2-1}{v}$. Separating: $\int \frac{v}{v^2-1}dv = \int \frac{dx}{x} \Rightarrow \frac{1}{2}\ln|v^2-1| = \ln|x| + C \Rightarrow v^2 - 1 = Ax^2$. Since $v = y/x$: $\frac{y^2}{x^2} - 1 = Ax^2 \Rightarrow y^2 = x^2 + Ax^4$. At $(1,2)$: $4 = 1 + A \Rightarrow A = 3$. So $y^2 = x^2 + 3x^4$. (b) Zero gradient requires the numerator $2v^2 - 1 = 0 \Rightarrow v = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$. So $m = \pm\frac{\sqrt{2}}{2}$.

5. Step 1: $y_1 = 1 + 0.1(0+1) = 1.1$. Step 2: $y_2 = 1.1 + 0.1(0.1+1.1) = 1.22$. Step 3: $y_3 = 1.22 + 0.1(0.2 + 1.22) = 1.362$. So $y(0.3) \approx 1.362$.

6. (a) Step 1 ($x = 1.25$): $y_1 = 2 + 0.25(1 + \frac{2}{1}) = 2.75$. Step 2 ($x = 1.5$): $y_2 = 2.75 + 0.25(1 + \frac{2.75}{1.25}) = 3.55$. Step 3 ($x = 1.75$): $y_3 = 3.55 + 0.25(1 + \frac{3.55}{1.5}) \approx 4.392$. Step 4 ($x = 2$): $y_4 \approx 4.392 + 0.25(1 + \frac{4.392}{1.75}) \approx 5.269$. So the Euler approximation gives $y(2) \approx 5.27$. (b) Let $y = vx$: $v + x\frac{dv}{dx} = 1 + v \Rightarrow x\frac{dv}{dx} = 1 \Rightarrow dv = \frac{dx}{x} \Rightarrow v = \ln x + C$. Since $v = y/x$: $y = x(\ln x + C)$. At $(1,2)$: $2 = 1(0 + C) \Rightarrow C = 2$. So $y = x(\ln x + 2)$. (c) $y(2) = 2(\ln 2 + 2) = 2\ln 2 + 4 \approx 5.386$. The Euler approximation (≈ 5.27) underestimates the actual value, since the curve is concave up over this interval and Euler's method uses the tangent line at each step.

7. (a) $T - 20 = 80e^{-kt}$. At $t = 5$, $T = 60$: $40 = 80e^{-5k} \Rightarrow e^{-5k} = 0.5 \Rightarrow k = \frac{\ln 2}{5} \approx 0.139$. (b) $T(10) = 20 + 80e^{-10k} = 20 + 80e^{-2\ln 2} = 20 + 80(0.25) = 40°$C.

8. (a) $\lambda = \frac{\ln 2}{6} \approx 0.116$. (b) $0.1 = e^{-\lambda t} \Rightarrow t = \frac{\ln 10}{\lambda} = \frac{6\ln 10}{\ln 2} \approx 19.9$ years.

9. $\int \frac{dy}{y} = \int \frac{x}{x^2+1}dx \Rightarrow \ln y = \frac{1}{2}\ln(x^2+1) + C \Rightarrow y = A\sqrt{x^2+1}$. At $(0,5)$: $5 = A$. So $y = 5\sqrt{x^2+1}$.

10. (a) $\frac{dP}{dt}$ represents the rate of change (growth rate) of the bacteria population with respect to time. (b) Differentiating $\frac{dP}{dt} = kP - \frac{kP^2}{N}$ with respect to t using the chain rule: $\frac{d^2P}{dt^2} = \left(k - \frac{2kP}{N}\right)\frac{dP}{dt} = k\left(1 - \frac{2P}{N}\right) \cdot kP\left(1 - \frac{P}{N}\right) = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$. (c) The growth rate $\frac{dP}{dt}$ is maximised where $\frac{d^2P}{dt^2} = 0$. From part (b), this occurs when $P = 0$, $P = N$, or $P = \frac{N}{2}$. Since $P = 0$ and $P = N$ both give $\frac{dP}{dt} = 0$ (minimum growth), the maximum must occur at $P = \frac{N}{2}$. Checking the sign of $\frac{d^2P}{dt^2}$ either side of $P = \frac{N}{2}$ confirms this is where $\frac{dP}{dt}$ changes from increasing to decreasing, confirming a maximum. (d) At $P = \frac{N}{2}$: $\frac{dP}{dt} = k\left(\frac{N}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{kN}{4}$. (e) Separating variables: $\int \frac{dP}{P(1-P/N)} = \int k\,dt$. Using partial fractions, $\frac{1}{P(1-P/N)} = \frac{1}{P} + \frac{1/N}{1-P/N} = \frac{1}{P} + \frac{1}{N-P}$. So $\ln P - \ln(N-P) = kt + C$. At $t = 0$, $P = P_0$: $C =$

$\ln P_0 - \ln(N - P_0)$. So $\ln \dfrac{P}{N-P} - \ln \dfrac{P_0}{N-P_0} = kt \Rightarrow kt = \ln\left[\dfrac{P}{P_0} \cdot \dfrac{N-P_0}{N-P}\right]$. (f) With $P = 2.5P_0$, $N = 5P_0$, $t = 8$: $8k = \ln\left[2.5 \cdot \dfrac{5P_0 - P_0}{5P_0 - 2.5P_0}\right] = \ln\left[2.5 \cdot \dfrac{4}{2.5}\right] = \ln 4$. So $k = \dfrac{\ln 4}{8} = \dfrac{\ln 2}{4} \approx 0.173$.