# Counting Principles

Worksheet 3 — Set C

*Mathematics: Analysis and Approaches HL*

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**Instructions**

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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**1. [Maximum mark: 3]** *Difficulty:* 1/5

In how many ways can 8 different flags be arranged in a row on a pole? **[3]**

**2. [Maximum mark: 4]** *Difficulty:* 1/5

A lock code consists of 5 digits, each chosen from 0 to 9, where digits may be repeated.

(a) Find the total number of possible codes. **[2]**

(b) Find the number of possible codes in which the first digit cannot be 0 and the last digit must be even. **[2]**

**3. [Maximum mark: 5]** *Difficulty:* 2/5

A committee of 5 people is to be chosen from a group of 12 people.

(a) Find the number of ways of forming the committee, with no restriction. **[2]**

(b) Find the number of ways of forming the committee if a particular person, Robin, must *not* be included. **[3]**

**4. [Maximum mark: 5]** *Difficulty:* 2/5

Consider the letters of the word **GARDEN**.

(a) Find the number of distinct arrangements of all six letters. **[2]**

(b) Find the number of these arrangements that start with a vowel. **[3]**

**5. [Maximum mark: 6]** *Difficulty:* 2/5

Consider the set of four-digit positive integers that can be formed from the digits $1,2,3,4,5,6,7,8,9$. Find the number of four-digit integers that can be formed such that

(a) the digits may be repeated; **[2]**

(b) the digits are distinct; **[2]**

(c) the digits are distinct and are in increasing order. **[2]**

**6. [Maximum mark: 7]** *Difficulty: 3/5*

A group of 5 friends and 4 family members is arranged in a row for a photograph, in which the 5 friends must all sit together.

(a) Find the number of ways to arrange the group with this restriction. **[4]**

(b) Find the number of ways in which, additionally, two particular friends within the group must not sit next to each other. **[3]**

**7. [Maximum mark: 7]** *Difficulty: 3/5*

A password consists of 6 characters, each chosen from the 62 characters formed by the lower-case letters, upper-case letters, and digits 0–9, where characters may be repeated.

(a) Find the total number of possible passwords. **[2]**

(b) Find the number of passwords in which no character is repeated. **[2]**

(c) Find the number of passwords that contain at least one repeated character. **[3]**

**8. [Maximum mark: 8]** *Difficulty: 4/5*

A farmer has seven sheep pens arranged in a single row. Four animals — Jax, Kim, Lee and Moe — are to be placed in the pens, with each pen holding at most one animal. Jax and Kim must not be placed in adjacent pens.

(a) Find the total number of ways to place the four animals into four of the seven pens, with no restriction. **[3]**

(b) Find the number of ways to place the four animals such that Jax and Kim are not in adjacent pens. **[5]**

**9. [Maximum mark: 8]** *Difficulty: 4/5*

Consider a three-digit code *abc*, where each of $a,b,c$ is assigned one of the values $1,2,3,4,5,6,7$, with no value repeated.

(a) Find the total number of possible codes. **[2]**

Let $N(x) = x^2 + ax + b$, where $a,b$ are two of the digits used in a code (with $a \neq b$). Given that $N(x)$ has a root at $x = -3$,

(b) find all possible pairs $(a,b)$ with $a,b \in \{1,2,3,4,5,6,7\}$. **[6]**

**10. [Maximum mark: 12]** *Difficulty: 5/5*

**Challenge question.** Consider a three-digit code *abc*, where each of $a,b,c$ is assigned one of the values $1,2,3,4,5,6,7$, with no value repeated.

(a) Find the total number of possible codes. **[2]**

Let $P(x) = x^3 + ax^2 + bx + c$, where $a,b,c$ are as assigned in the code above. Suppose $P(x)$ has a factor of $(x^2 + 2x + 5)$.

(b) By writing $P(x) = (x^2 + 2x + 5)(x + k)$ for some constant $k$, find expressions for $a$, $b$ and $c$ in terms of $k$. **[4]**

(c) Hence, using the constraint that $a,b,c \in \{1,2,3,4,5,6,7\}$ with no value repeated, show that the only possible assignment is $a = 3$, $b = 7$, $c = 5$. **[4]**

(d) Express $P(x)$ as a product of a linear factor and an irreducible quadratic factor. **[2]**

# Answer Key

**1.** $8! = 40\,320$.

**2.** (a) $10^5 = 100\,000$. (b) First digit: 9 choices (cannot be 0); last digit: 5 choices (must be $0, 2, 4, 6$ or 8); middle three digits: 10 each. Total $= 9 \times 10 \times 10 \times 10 \times 5 = 45\,000$.

**3.** (a) $\binom{12}{5} = 792$. (b) With Robin excluded, choose all 5 from the remaining 11: $\binom{11}{5} = 462$.

**4.** (a) $6! = 720$. (b) GARDEN has vowels A, E (2 vowels). First letter: 2 choices; remaining 5 letters: $5! = 120$ arrangements. Total $= 2 \times 120 = 240$.

**5.** (a) $9^4 = 6561$. (b) ${}^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$. (c) $\binom{9}{4} = 126$.

**6.** (a) Treat the 5 friends as one block: 1 block $+4$ family members $= 5$ units, arranged in $5!$ ways; the friends within the block can be arranged in $5!$ ways. Total $= 5! \times 5! = 120 \times 120 = 14\,400$. (b) Within the block, total arrangements of 5 friends $= 5! = 120$. Arrangements with the two particular friends adjacent: treat them as one unit with the other 3 friends (4 units, $4! = 24$ arrangements) $\times 2$ (internal order) $= 48$. Not-adjacent arrangements $= 120 - 48 = 72$. Total $= 5! \times 72 = 120 \times 72 = 8640$.

**7.** (a) $62^6 = 56\,800\,235\,584$. (b) ${}^{62}P_6 = 62 \times 61 \times 60 \times 59 \times 58 \times 57 = 44\,261\,653\,680$. (c) At least one repeat $= 56\,800\,235\,584 - 44\,261\,653\,680 = 12\,538\,581\,904$.

**8.** (a) ${}^7P_4 = 7 \times 6 \times 5 \times 4 = 840$. (b) Number of adjacent pairs of pens in a row of 7: 6. For each pair, Jax and Kim can be ordered in 2 ways, and the remaining 2 animals placed in 2 of the remaining 5 pens in ${}^5P_2 = 20$ ways. Adjacent total $= 6 \times 2 \times 20 = 240$. Not-adjacent total $= 840 - 240 = 600$.

**9.** (a) ${}^7P_3 = 7 \times 6 \times 5 = 210$. (b) $N(-3) = 9 - 3a + b = 0 \Rightarrow b = 3a - 9$. Testing $a = 1, \ldots, 7$: $a = 4$ gives $b = 3$ (valid); $a = 5$ gives $b = 6$ (valid); $a = 3$ gives $b = 0$ (invalid, out of range); $a = 6$ gives $b = 9$ (invalid, out of range); $a = 7$ gives $b = 12$ (invalid, out of range). So the possible pairs are $(4, 3)$ and $(5, 6)$.

**10.** (a) ${}^7P_3 = 7 \times 6 \times 5 = 210$. (b) Expanding $(x^2 + 2x + 5)(x + k) = x^3 + (k + 2)x^2 + (2k + 5)x + 5k$, so $a = k + 2$, $b = 2k + 5$, $c = 5k$. (c) Testing integer values of $k$: $k = 1$ gives $a = 3, b = 7, c = 5$, all in $\{1, \ldots, 7\}$ and distinct — valid. $k = 0$ gives $c = 0$ (invalid, out of range). $k = 2$ gives $a = 4, b = 9$ (invalid, $b$ out of range). No other $k$ works. So the only possible assignment is $a = 3, b = 7, c = 5$. (d) $P(x) = x^3 + 3x^2 + 7x + 5 = (x^2 + 2x + 5)(x + 1)$. Since $x^2 + 2x + 5$ has discriminant $4 - 20 = -16 < 0$, it is irreducible over $\mathbb{R}$, so this is the required factorisation.