Counting Principles

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

In how many ways can 7 different photographs be arranged in a row in an album? **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

A PIN consists of 4 digits, each chosen from 0 to 9, where digits may be repeated.

(a) Find the total number of possible PINs. **[2]**

(b) Find the number of possible PINs in which the first digit is not 0 and the last digit is not 0. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

A committee of 4 people is to be chosen from a group of 10 people.

(a) Find the number of ways of forming the committee, with no restriction. **[2]**

(b) Find the number of ways of forming the committee if a particular person, Jamie, must be included. **[3]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Consider the letters of the word **PLANET**.

(a) Find the number of distinct arrangements of all six letters. **[2]**

(b) Find the number of these arrangements that end in a consonant. **[3]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Consider the set of five-digit positive integers that can be formed from the digits $1, 2, 3, 4, 5, 6, 7, 8$. Find the number of five-digit integers that can be formed such that

(a) the digits may be repeated; **[2]**

(b) the digits are distinct; **[2]**

(c) the digits are distinct and are in increasing order. **[2]**

6. [Maximum mark: 7] *Difficulty:* 3/5

A group of 3 girls and 6 boys is arranged in a row for a photograph, in which the 3 girls must all sit together.

(a) Find the number of ways to arrange the group with this restriction. **[4]**

(b) Find the number of ways in which, additionally, two particular girls within the group must not sit next to each other. **[3]**

7. [Maximum mark: 7] *Difficulty:* 3/5

A password consists of 4 characters, each chosen from the 36 characters formed by the 26 letters of the alphabet together with the 10 digits 0–9, where characters may be repeated.

(a) Find the total number of possible passwords. **[2]**

(b) Find the number of passwords in which no character is repeated. **[2]**

(c) Find the number of passwords that contain at least one repeated character. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

A farmer has eight sheep pens arranged in a single row. Five animals — Eve, Fay, Gus, Hal and Ivy — are to be placed in the pens, with each pen holding at most one animal. Eve and Fay must not be placed in adjacent pens.

(a) Find the total number of ways to place the five animals into five of the eight pens, with no restriction. **[3]**

(b) Find the number of ways to place the five animals such that Eve and Fay are not in adjacent pens. **[5]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Consider a three-digit code abc, where each of a,b,c is assigned one of the values $1,2,3,4,5,6$, with no value repeated.

(a) Find the total number of possible codes. **[2]**

Let $N(x) = x^2 + ax + b$, where a,b are two of the digits used in a code (with $a \neq b$). Given that $N(x)$ has a root at $x = -2$,

(b) find all possible pairs (a,b) with $a,b \in \{1,2,3,4,5,6\}$. **[6]**

10. [Maximum mark: 12] *Difficulty:* 5/5

Challenge question. Consider a three-digit code abc, where each of a,b,c is assigned one of the values $1,2,3,4,5,6$, with no value repeated.

(a) Find the total number of possible codes. **[2]**

Let $P(x) = x^3 + ax^2 + bx + c$, where a,b,c are as assigned in the code above. Suppose $P(x)$ has a factor of $(x^2 + x + 2)$.

(b) By writing $P(x) = (x^2 + x + 2)(x + k)$ for some constant k, find expressions for a, b and c in terms of k. **[4]**

(c) Hence, using the constraint that $a,b,c \in \{1,2,3,4,5,6\}$ with no value repeated, show that the only possible assignment is $a = 4$, $b = 5$, $c = 6$. **[4]**

(d) Express $P(x)$ as a product of a linear factor and an irreducible quadratic factor. **[2]**

Answer Key

1. $7! = 5040$.

2. (a) $10^4 = 10\,000$. (b) First digit: 9 choices; last digit: 9 choices; middle two digits: 10 each. Total $= 9 \times 10 \times 10 \times 9 = 8100$.

3. (a) $\binom{10}{4} = 210$. (b) With Jamie included, choose the remaining 3 from the other 9: $\binom{9}{3} = 84$.

4. (a) $6! = 720$. (b) PLANET has consonants P, L, N, T (4 of the 6 letters). Last letter: 4 choices; remaining 5 letters: $5! = 120$ arrangements. Total $= 4 \times 120 = 480$.

5. (a) $8^5 = 32\,768$. (b) ${}^8P_5 = 8 \times 7 \times 6 \times 5 \times 4 = 6720$. (c) $\binom{8}{5} = 56$.

6. (a) Treat the 3 girls as one block: 1 block $+6$ boys $= 7$ units, arranged in $7!$ ways; the girls within the block can be arranged in $3!$ ways. Total $= 7! \times 3! = 5040 \times 6 = 30\,240$. (b) Within the block, total arrangements of 3 girls $= 3! = 6$. Arrangements with the two particular girls adjacent: treat them as one unit with the remaining girl (2 units, $2! = 2$ arrangements) $\times 2$ (internal order) $= 4$. Not-adjacent arrangements $= 6 - 4 = 2$. Total $= 7! \times 2 = 5040 \times 2 = 10\,080$.

7. (a) $36^4 = 1\,679\,616$. (b) ${}^{36}P_4 = 36 \times 35 \times 34 \times 33 = 1\,413\,720$. (c) At least one repeat $= 1\,679\,616 - 1\,413\,720 = 265\,896$.

8. (a) ${}^8P_5 = 8 \times 7 \times 6 \times 5 \times 4 = 6720$. (b) Number of adjacent pairs of pens in a row of 8: 7. For each pair, Eve and Fay can be ordered in 2 ways, and the remaining 3 animals placed in 3 of the remaining 6 pens in ${}^6P_3 = 120$ ways. Adjacent total $= 7 \times 2 \times 120 = 1680$. Not-adjacent total $= 6720 - 1680 = 5040$.

9. (a) ${}^6P_3 = 6 \times 5 \times 4 = 120$. (b) $N(-2) = 4 - 2a + b = 0 \Rightarrow b = 2a - 4$. Testing $a = 1, \ldots, 6$: $a = 3$ gives $b = 2$ (valid); $a = 4$ gives $b = 4$ (invalid, $a = b$); $a = 5$ gives $b = 6$ (valid); $a = 2$ gives $b = 0$ (invalid, out of range); $a = 6$ gives $b = 8$ (invalid, out of range). So the possible pairs are $(3,2)$ and $(5,6)$.

10. (a) ${}^6P_3 = 6 \times 5 \times 4 = 120$. (b) Expanding $(x^2 + x + 2)(x + k) = x^3 + (k+1)x^2 + (k+2)x + 2k$, so $a = k+1$, $b = k+2$, $c = 2k$. (c) Testing integer values of k: $k = 1$ gives $a = 2, b = 3, c = 2$ (invalid, $a = c$). $k = 2$ gives $a = 3, b = 4, c = 4$ (invalid, $b = c$). $k = 3$ gives $a = 4, b = 5, c = 6$, all in $\{1, \ldots, 6\}$ and distinct — valid. $k = 4$ gives $b = 6, c = 8$ (invalid, c out of range). No other k works. So the only possible assignment is $a = 4, b = 5, c = 6$. (d) $P(x) = x^3 + 4x^2 + 5x + 6 = (x^2 + x + 2)(x + 3)$. Since $x^2 + x + 2$ has discriminant $1 - 8 = -7 < 0$, it is irreducible over $\mathbb{R}$, so this is the required factorisation.