Counting Principles

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

In how many ways can 6 different toy cars be arranged in a row on a shelf? **[3]**

2. [Maximum mark: 4] *Difficulty:* 1/5

A code consists of 4 digits, each chosen from 0 to 9, where digits may be repeated.

(a) Find the number of possible codes with no restriction. **[2]**

(b) Find the number of possible codes if the first digit cannot be 0. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

A committee of 3 people is to be chosen from a group of 8 people.

(a) Find the number of ways of forming the committee, with no restriction. **[2]**

(b) Find the number of ways of forming the committee if a particular person, Alex, must be included. **[3]**

4. [Maximum mark: 5] *Difficulty:* 2/5

Consider the letters of the word **NUMBER**.

(a) Find the number of distinct arrangements of all six letters. **[2]**

(b) Find the number of these arrangements that start with a vowel. **[3]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Consider the set of five-digit positive integers that can be formed from the digits $1, 2, 3, 4, 5, 6, 7$. Find the number of five-digit integers that can be formed such that

(a) the digits may be repeated; **[2]**

(b) the digits are distinct; **[2]**

(c) the digits are distinct and are in increasing order. **[2]**

6. [Maximum mark: 7] *Difficulty:* 3/5

A group of 4 boys and 5 girls is arranged in a row for a photograph, in which the 4 boys must all sit together.

(a) Find the number of ways to arrange the group with this restriction. **[4]**

(b) Find the number of ways in which, additionally, two particular boys within the group must not sit next to each other. **[3]**

7. [Maximum mark: 7] *Difficulty:* 3/5

A password consists of 5 characters, each chosen from the 26 letters of the alphabet, where letters may be repeated.

(a) Find the total number of possible passwords. **[2]**

(b) Find the number of passwords in which no letter is repeated. **[2]**

(c) Find the number of passwords that contain at least one repeated letter. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

A farmer has six sheep pens arranged in a single row. Four sheep — Ada, Bo, Cy and Dee — are to be placed in the pens, with each pen holding at most one sheep. Ada and Bo are known to fight and must not be placed in adjacent pens.

(a) Find the total number of ways to place the four sheep into four of the six pens, with no restriction. **[3]**

(b) Find the number of ways to place the four sheep such that Ada and Bo are not in adjacent pens. **[5]**

9. [Maximum mark: 8] *Difficulty:* 4/5

Consider a three-digit code *abc*, where each of a,b,c is assigned one of the values $1,2,3,4,5$.

(a) Find the total number of possible codes, assuming each value can be repeated. **[2]**

(b) Find the total number of possible codes, assuming no value is repeated. **[2]**

Let $N(x) = x^2 + ax + b$, where a,b are two of the digits used in a code (with $a \neq b$). Given that $N(x)$ has a root at $x = -1$,

(c) find all possible pairs (a,b) with $a,b \in \{1,2,3,4,5\}$. **[4]**

10. [Maximum mark: 12] *Difficulty:* 5/5

Challenge question. Consider a three-digit code *abc*, where each of a,b,c is assigned one of the values $1,2,3,4,5$, with no value repeated.

(a) Find the total number of possible codes. **[2]**

Let $P(x) = x^3 + ax^2 + bx + c$, where a,b,c are as assigned in the code above. Suppose $P(x)$ has a factor of $(x^2 + 2x + 2)$.

(b) By writing $P(x) = (x^2 + 2x + 2)(x + k)$ for some constant k, find expressions for a, b and c in terms of k. **[4]**

(c) Hence, using the constraint that $a,b,c \in \{1,2,3,4,5\}$ with no value repeated, show that the only possible assignment is $a = 3$, $b = 4$, $c = 2$. **[4]**

(d) Express $P(x)$ as a product of a linear factor and an irreducible quadratic factor. **[2]**

Answer Key

1. $6! = 720$.

2. (a) $10^4 = 10\,000$. (b) $9 \times 10^3 = 9000$ (the first digit has 9 choices, the rest 10 each).

3. (a) $\binom{8}{3} = 56$. (b) With Alex included, choose the remaining 2 from the other 7: $\binom{7}{2} = 21$.

4. (a) $6! = 720$. (b) NUMBER has vowels U, E (2 vowels). First letter: 2 choices; remaining 5 letters: $5! = 120$ arrangements. Total $= 2 \times 120 = 240$.

5. (a) $7^5 = 16\,807$. (b) ${}^7P_5 = \dfrac{7!}{2!} = 2520$. (c) $\binom{7}{5} = 21$ (each set of 5 distinct digits has exactly one increasing arrangement).

6. (a) Treat the 4 boys as one block: 1 block $+5$ girls $= 6$ units, arranged in $6!$ ways; the boys within the block can be arranged in $4!$ ways. Total $= 6! \times 4! = 720 \times 24 = 17\,280$. (b) Within the block, total arrangements of the 4 boys $= 4! = 24$. Arrangements with the two particular boys adjacent: treat them as one unit with the other 2 boys (3 units, $3! = 6$ arrangements) $\times 2$ (internal order) $= 12$. So arrangements with them *not* adjacent $= 24 - 12 = 12$. Total $= 6! \times 12 = 720 \times 12 = 8640$.

7. (a) $26^5 = 11\,881\,376$. (b) ${}^{26}P_5 = 26 \times 25 \times 24 \times 23 \times 22 = 7\,893\,600$. (c) At least one repeat $= 11\,881\,376 - 7\,893\,600 = 3\,987\,776$.

8. (a) ${}^6P_4 = 6 \times 5 \times 4 \times 3 = 360$. (b) Number of adjacent pairs of pens in a row of 6: 5 (pairs $(1,2),\ldots,(5,6)$). For each pair, Ada and Bo can be ordered in 2 ways, and the remaining 2 sheep placed in 2 of the remaining 4 pens in ${}^4P_2 = 12$ ways. Adjacent total $= 5 \times 2 \times 12 = 120$. Not-adjacent total $= 360 - 120 = 240$.

9. (a) $5^3 = 125$. (b) ${}^5P_3 = 5 \times 4 \times 3 = 60$. (c) $N(-1) = 1 - a + b = 0 \Rightarrow b = a - 1$. Testing $a = 1,\ldots,5$: only $a = 2,3,4,5$ give $b = 1,2,3,4$ respectively, all valid (and distinct from a). So the possible pairs are $(2,1),(3,2),(4,3),(5,4)$.

10. (a) ${}^5P_3 = 5 \times 4 \times 3 = 60$. (b) Expanding $(x^2+2x+2)(x+k) = x^3 + (k+2)x^2 + (2k+2)x + 2k$, so $a = k+2, b = 2k+2, c = 2k$. (c) Testing integer values of k: $k = 1$ gives $a = 3, b = 4, c = 2$, all in $\{1,\ldots,5\}$ and distinct — valid. $k = 0$ gives $c = 0$ (invalid, out of range). $k = 2$ gives $b = 6$ (invalid, out of range). No other k gives all of a,b,c in range and distinct. So the only possible assignment is $a = 3, b = 4, c = 2$. (d) $P(x) = x^3 + 3x^2 + 4x + 2 = (x^2+2x+2)(x+1)$. Since $x^2 + 2x + 2$ has discriminant $4 - 8 = -4 < 0$, it is irreducible over $\mathbb{R}$, so this is the required factorisation.