Complex Numbers

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Let $z_1 = -3+5i$ and $z_2 = 2-i$.

(a) Find $z_1 - z_2$. **[1]**

(b) Find $z_1 z_2$. **[2]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Express $z = -\sqrt{3}+i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$, giving r and θ exactly.

(a) Find r. **[2]**

(b) Find θ. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Solve the equation $z^2 - 2z + 10 = 0$, giving your answers in the form $a+bi$.

(a) Use the quadratic formula and simplify the discriminant. **[3]**

(b) State the two roots. **[2]**

4. [Maximum mark: 5] *Difficulty:* 2/5

A cubic equation with real coefficients has $2+i$ and 3 among its roots.

(a) Write down another root of the equation. **[1]**

(b) Hence find the cubic equation, in the form $z^3 + az^2 + bz + c = 0$. **[4]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Use De Moivre's theorem to find $(\sqrt{3}+i)^6$, giving your answer in the form $a+bi$.

(a) Express $\sqrt{3}+i$ in the form $r(\cos\theta + i\sin\theta)$. **[2]**

(b) Hence apply De Moivre's theorem and simplify. **[4]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the equation $z^4 + pz^3 + qz^2 + rz + 135 = 0$, where $z \in \mathbb{C}$ and $p, q, r \in \mathbb{R}$. Three of the roots of the equation are $1+2i$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all four roots of the equation, find the value of α. **[4]**

(b) Hence, by considering the sum of all four roots, find the value of p. **[3]**

7. [Maximum mark: 6] *Difficulty:* 3/5

Consider the complex numbers $z_1 = 4 + bi$ and $z_2 = (16 - b^2) - 8bi$, where $b \in \mathbb{R}$, $b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. **[3]**

(b) Hence, given that $\arg(z_1 z_2) = \dfrac{\pi}{4}$, find the value of b. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Let z_1 and z_2 be non-zero complex numbers such that $\mathrm{Re}(z_1 \overline{z_2}) = 0$.

(a) Show that this condition implies that the position vectors of z_1 and z_2 in the Argand diagram are perpendicular. **[4]**

(b) Given that $z_1 = 8 + 6i$ and $\mathrm{Re}(z_1 \overline{z_2}) = 0$, find the general form of z_2 in terms of a real parameter $k \neq 0$. **[4]**

9. [Maximum mark: 9] *Difficulty:* 4/5

Consider the quartic equation $z^4 - 14z^3 + 98z^2 - 350z + 625 = 0$, where $z \in \mathbb{C}$. Two of the roots are $a + bi$ and $b + ai$, where $a, b \in \mathbb{Z}$.

(a) By considering the sum of the four roots, show that $a + b = 7$. **[3]**

(b) By considering the product of all four roots, find the value of $a^2 + b^2$. **[3]**

(c) Hence find the possible values of a. **[3]**

10. [Maximum mark: 13] *Difficulty:* 5/5

Challenge question. Points Z_1 and Z_2 in the Argand diagram represent non-zero complex numbers z_1 and z_2 respectively. Triangle OZ_1Z_2 is equilateral, such that $z_2 = z_1 e^{-i\pi/3}$.

(a) Show that $z_1^2 + z_2^2 = z_1 z_2$. **[6]**

(b) Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$, where $a, b \in \mathbb{C}$. Using the result from part (a) and the relationships between roots and coefficients, show that $a^2 = 3b$. **[5]**

(c) Hence, given that $a, b \in \mathbb{R}$ and $a = 3\sqrt{3}$, find the value of b. **[2]**

Answer Key

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1. (a) $z_1 - z_2 = -5+6i$. (b) $z_1z_2 = (-3+5i)(2-i) = -6+3i+10i-5i^2 = -6+13i+5 = -1+13i$.

2. (a) $r = \sqrt{(\sqrt{3})^2+1^2} = 2$. (b) Reference angle $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$; since z lies in the second quadrant, $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

3. (a) $z = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2}$. (b) $z = 1+3i$ or $z = 1-3i$.

4. (a) Since coefficients are real, $2-i$ is also a root. (b) Sum of roots $= (2+i)+(2-i)+3 = 7 \Rightarrow a = -7$. Sum of products of pairs: $(2+i)(2-i)+(2+i)(3)+(2-i)(3) = 5+12 = 17 \Rightarrow b = 17$. Product of roots $= 5(3) = 15 \Rightarrow c = -15$. Equation: $z^3 - 7z^2 + 17z - 15 = 0$.

5. (a) $\sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. (b) $(\sqrt{3}+i)^6 = 2^6(\cos\pi + i\sin\pi) = 64(-1+0i) = -64$.

6. (a) Product of conjugate pair: $(1+2i)(1-2i) = 5$. Product of all roots $= 135$ (constant term), so $\alpha^3 = 135/5 = 27 \Rightarrow \alpha = 3$. (b) Sum of roots $= (1+2i)+(1-2i)+3+9 = 14 = -p \Rightarrow p = -14$.

7. (a) $z_1z_2 = (4+bi)\left[(16-b^2)-8bi\right] = (64+4b^2)-(b^3+16b)i$. (b) Since $z_2 = \overline{z_1^2}$, $z_1z_2 = |z_1|^2\overline{z_1}$, so $\arg(z_1z_2) = -\arg(z_1) = -\arctan\left(\frac{b}{4}\right)$. Setting this equal to $\frac{\pi}{4}$: $\frac{b}{4} = -1 \Rightarrow b = -4$.

8. (a) As before: $\text{Re}(z_1\overline{z_2})$ is the dot product of the position vectors of z_1 and z_2, which is zero exactly when the (non-zero) vectors are perpendicular. (b) Require $8x_2+6y_2 = 0 \Rightarrow x_2 = -\frac{3y_2}{4}$. Letting $y_2 = 4k$, $x_2 = -3k$: $z_2 = k(-3+4i)$, $k \in \mathbb{R}\setminus\{0\}$.

9. (a) Sum of roots $= 2a+2b = 14 \Rightarrow a+b = 7$. (b) Product of roots $= 625$; since the roots pair to give $(a^2+b^2)^2 = 625$, $a^2+b^2 = 25$. (c) $(a+b)^2 = 49 = 25+2ab \Rightarrow ab = 12$. So a,b are roots of $t^2-7t+12 = 0 \Rightarrow (t-3)(t-4) = 0$, giving $a = 3$ or $a = 4$.

10. (a) Since $z_2 = z_1e^{-i\pi/3}$, $z_2^2 = z_1^2e^{-i2\pi/3}$, so $z_1^2+z_2^2 = z_1^2\left(1+e^{-i2\pi/3}\right)$. Now $e^{-i2\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, so $1+e^{-i2\pi/3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-i\pi/3}$. Hence $z_1^2+z_2^2 = z_1^2e^{-i\pi/3} = z_1(z_1e^{-i\pi/3}) = z_1z_2$. (b) By Vieta's formulas, $z_1+z_2 = -a$, $z_1z_2 = b$, so $z_1^2+z_2^2 = (z_1+z_2)^2-2z_1z_2 = a^2-2b$. By part (a) this equals b, so $a^2-2b = b \Rightarrow a^2 = 3b$. (c) With $a = 3\sqrt{3}$: $a^2 = 27 = 3b \Rightarrow b = 9$.