Complex Numbers

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Let $z_1 = 4 - i$ and $z_2 = -2 + 3i$.

(a) Find $z_1 + z_2$. **[1]**

(b) Find $z_1 z_2$. **[2]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Express $z = -1 + i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$, giving r and θ exactly.

(a) Find r. **[2]**

(b) Find θ. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Solve the equation $z^2 - 6z + 13 = 0$, giving your answers in the form $a + bi$.

(a) Use the quadratic formula and simplify the discriminant. **[3]**

(b) State the two roots. **[2]**

4. [Maximum mark: 5] *Difficulty:* 2/5

A cubic equation with real coefficients has $1 - 2i$ and -1 among its roots.

(a) Write down another root of the equation. **[1]**

(b) Hence find the cubic equation, in the form $z^3 + az^2 + bz + c = 0$. **[4]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Use De Moivre's theorem to find $(1 - i)^6$, giving your answer in the form $a + bi$.

(a) Express $1 - i$ in the form $r(\cos\theta + i\sin\theta)$. **[2]**

(b) Hence apply De Moivre's theorem and simplify. **[4]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the equation $z^4 + pz^3 + qz^2 + rz + 128 = 0$, where $z \in \mathbb{C}$ and $p, q, r \in \mathbb{R}$. Three of the roots of the equation are $1+i$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all four roots of the equation, find the value of α. **[4]**

(b) Hence, by considering the sum of all four roots, find the value of p. **[3]**

7. [Maximum mark: 6] *Difficulty:* 3/5

Consider the complex numbers $z_1 = 3 + bi$ and $z_2 = (9 - b^2) - 6bi$, where $b \in \mathbb{R}$, $b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. **[3]**

(b) Hence, given that $\arg(z_1 z_2) = \dfrac{\pi}{4}$, find the value of b. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Let z_1 and z_2 be non-zero complex numbers such that $\mathrm{Re}(z_1 \overline{z_2}) = 0$.

(a) Show that this condition implies that the position vectors of z_1 and z_2 in the Argand diagram are perpendicular. **[4]**

(b) Given that $z_1 = 5 + 12i$ and $\mathrm{Re}(z_1 \overline{z_2}) = 0$, find the general form of z_2 in terms of a real parameter $k \neq 0$. **[4]**

9. [Maximum mark: 9] *Difficulty:* 4/5

Consider the quartic equation $z^4 - 16z^3 + 128z^2 - 640z + 1600 = 0$, where $z \in \mathbb{C}$. Two of the roots are $a + bi$ and $b + ai$, where $a, b \in \mathbb{Z}$.

(a) By considering the sum of the four roots, show that $a + b = 8$. **[3]**

(b) By considering the product of all four roots, find the value of $a^2 + b^2$. **[3]**

(c) Hence find the possible values of a. **[3]**

10. [Maximum mark: 13] *Difficulty:* 5/5

Challenge question. Points Z_1 and Z_2 in the Argand diagram represent non-zero complex numbers z_1 and z_2 respectively. Triangle OZ_1Z_2 is equilateral, such that $z_2 = z_1 e^{i\pi/3}$.

(a) Show that $z_1^2 + z_2^2 = z_1 z_2$. **[6]**

(b) Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$, where $a, b \in \mathbb{C}$. Using the result from part (a) and the relationships between roots and coefficients, show that $a^2 = 3b$. **[5]**

(c) Hence, given that $a, b \in \mathbb{R}$ and $a = 9$, find the value of b. **[2]**

Answer Key

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1. (a) $z_1 + z_2 = 2 + 2i$. (b) $z_1 z_2 = (4 - i)(-2 + 3i) = -8 + 12i + 2i - 3i^2 = -8 + 14i + 3 = -5 + 14i$.

2. (a) $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. (b) Reference angle $\arctan(1/1) = \pi/4$; since z lies in the second quadrant, $\theta = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$.

3. (a) $z = \dfrac{6 \pm \sqrt{36 - 52}}{2} = \dfrac{6 \pm \sqrt{-16}}{2} = \dfrac{6 \pm 4i}{2}$. (b) $z = 3 + 2i$ or $z = 3 - 2i$.

4. (a) Since coefficients are real, $1 + 2i$ is also a root. (b) Sum of roots $= (1 + 2i) + (1 - 2i) + (-1) = 1 \Rightarrow a = -1$. Sum of products of pairs: $(1+2i)(1-2i) + (1+2i)(-1) + (1-2i)(-1) = 5 - 2 = 3 \Rightarrow b = 3$. Product of roots $= 5(-1) = -5 \Rightarrow c = 5$. Equation: $z^3 - z^2 + 3z + 5 = 0$.

5. (a) $1 - i = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$. (b) $(1 - i)^6 = (\sqrt{2})^6\left(\cos\left(-\dfrac{3\pi}{2}\right) + i\sin\left(-\dfrac{3\pi}{2}\right)\right) = 8(0 + 1i) = 8i$.

6. (a) Product of conjugate pair: $(1 + i)(1 - i) = 2$. Product of all roots $= 128$ (constant term), so $\alpha^3 = 128/2 = 64 \Rightarrow \alpha = 4$. (b) Sum of roots $= (1 + i) + (1 - i) + 4 + 16 = 22 = -p \Rightarrow p = -22$.

7. (a) $z_1 z_2 = (3 + bi)\left[(9 - b^2) - 6bi\right] = (27 + 3b^2) - (b^3 + 9b)i$. (b) Since $z_2 = \overline{z_1^2}$, $z_1 z_2 = |z_1|^2\overline{z_1}$, so $\arg(z_1 z_2) = -\arg(z_1) = -\arctan\left(\dfrac{b}{3}\right)$. Setting this equal to $\dfrac{\pi}{4}$: $\dfrac{b}{3} = -1 \Rightarrow b = -3$.

8. (a) As in Worksheet A: $\mathrm{Re}(z_1\overline{z_2})$ is the dot product of the position vectors of z_1 and z_2, which is zero exactly when the (non-zero) vectors are perpendicular. (b) Require $5x_2 + 12y_2 = 0 \Rightarrow x_2 = -\dfrac{12y_2}{5}$. Letting $y_2 = 5k$, $x_2 = -12k$: $z_2 = k(-12 + 5i)$, $k \in \mathbb{R} \setminus \{0\}$.

9. (a) Sum of roots $= 2a + 2b = 16 \Rightarrow a + b = 8$. (b) Product of roots $= 1600$; since the roots pair to give $(a^2 + b^2)^2 = 1600$, $a^2 + b^2 = 40$. (c) $(a + b)^2 = 64 = 40 + 2ab \Rightarrow ab = 12$. So a, b are roots of $t^2 - 8t + 12 = 0 \Rightarrow (t - 2)(t - 6) = 0$, giving $a = 2$ or $a = 6$.

10. (a) Since $z_2 = z_1 e^{i\pi/3}$, $z_2^2 = z_1^2 e^{i2\pi/3}$, so $z_1^2 + z_2^2 = z_1^2\left(1 + e^{i2\pi/3}\right)$. Now $e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, so $1 + e^{i2\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\pi/3}$. Hence $z_1^2 + z_2^2 = z_1^2 e^{i\pi/3} = z_1(z_1 e^{i\pi/3}) = z_1 z_2$. (b) By Vieta's formulas, $z_1 + z_2 = -a$, $z_1 z_2 = b$, so $z_1^2 + z_2^2 = (z_1 + z_2)^2 - 2z_1 z_2 = a^2 - 2b$. By part (a) this equals b, so $a^2 - 2b = b \Rightarrow a^2 = 3b$. (c) With $a = 9$: $81 = 3b \Rightarrow b = 27$.