Complex Numbers

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 3] *Difficulty:* 1/5

Let $z_1 = 3 + 2i$ and $z_2 = 1 - 4i$.

(a) Find $z_1 + z_2$. **[1]**

(b) Find $z_1 z_2$. **[2]**

2. [Maximum mark: 4] *Difficulty:* 1/5

Express $z = 1 + i\sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$, giving r and θ exactly.

(a) Find r. **[2]**

(b) Find θ. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Solve the equation $z^2 - 4z + 13 = 0$, giving your answers in the form $a + bi$.

(a) Use the quadratic formula and simplify the discriminant. **[3]**

(b) State the two roots. **[2]**

4. [Maximum mark: 5] *Difficulty:* 2/5

A cubic equation with real coefficients has $2 + 3i$ and 1 among its roots.

(a) Write down another root of the equation. **[1]**

(b) Hence find the cubic equation, in the form $z^3 + az^2 + bz + c = 0$. **[4]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Use De Moivre's theorem to find $(1 + i)^8$, giving your answer in the form $a + bi$.

(a) Express $1 + i$ in the form $r(\cos\theta + i\sin\theta)$. **[2]**

(b) Hence apply De Moivre's theorem and simplify. **[4]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Consider the equation $z^4 + pz^3 + qz^2 + rz + 320 = 0$, where $z \in \mathbb{C}$ and $p, q, r \in \mathbb{R}$. Three of the roots of the equation are $2+i$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all four roots of the equation, find the value of α. **[4]**

(b) Hence, by considering the sum of all four roots, find the value of p. **[3]**

7. [Maximum mark: 6] *Difficulty: 3/5*

Consider the complex numbers $z_1 = 2 + bi$ and $z_2 = (4 - b^2) - 4bi$, where $b \in \mathbb{R}$, $b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. **[3]**

(b) Hence, given that $\arg(z_1 z_2) = \dfrac{\pi}{4}$, find the value of b. **[3]**

8. [Maximum mark: 8] *Difficulty: 4/5*

Let z_1 and z_2 be non-zero complex numbers such that $\text{Re}(z_1 \overline{z_2}) = 0$.

(a) Show that this condition implies that the position vectors of z_1 and z_2 in the Argand diagram are perpendicular. **[4]**

(b) Given that $z_1 = 3 + 4i$ and $\text{Re}(z_1 \overline{z_2}) = 0$, find the general form of z_2 in terms of a real parameter $k \neq 0$. **[4]**

9. [Maximum mark: 9] *Difficulty: 4/5*

Consider the quartic equation $z^4 - 8z^3 + 32z^2 - 80z + 100 = 0$, where $z \in \mathbb{C}$. Two of the roots are $a + bi$ and $b + ai$, where $a, b \in \mathbb{Z}$.

(a) By considering the sum of the four roots, show that $a + b = 4$. **[3]**

(b) By considering the product of all four roots, find the value of $a^2 + b^2$. **[3]**

(c) Hence find the possible values of a. **[3]**

10. [Maximum mark: 13] *Difficulty: 5/5*

Challenge question. Points Z_1 and Z_2 in the Argand diagram represent non-zero complex numbers z_1 and z_2 respectively. Triangle OZ_1Z_2 is equilateral, such that $z_2 = z_1 e^{-i\pi/3}$.

(a) Show that $z_1^2 + z_2^2 = z_1 z_2$. **[6]**

(b) Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$, where $a, b \in \mathbb{C}$. Using the result from part (a) and the relationships between roots and coefficients, show that $a^2 = 3b$. **[5]**

(c) Hence, given that $a, b \in \mathbb{R}$ and $a = 6$, find the value of b. **[2]**

Answer Key

1. (a) $z_1 + z_2 = 4 - 2i$. (b) $z_1 z_2 = (3+2i)(1-4i) = 3 - 12i + 2i - 8i^2 = 3 - 10i + 8 = 11 - 10i$.

2. (a) $r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$. (b) $\theta = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$ (first quadrant).

3. (a) $z = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}$. (b) $z = 2 + 3i$ or $z = 2 - 3i$.

4. (a) Since coefficients are real, $2 - 3i$ is also a root. (b) Sum of roots $= (2+3i) + (2-3i) + 1 = 5 \Rightarrow a = -5$. Sum of products of pairs $= (2+3i)(2-3i) + (2+3i)(1) + (2-3i)(1) = 13 + 4 = 17 \Rightarrow b = 17$. Product of roots $= 13(1) = 13 \Rightarrow c = -13$. Equation: $z^3 - 5z^2 + 17z - 13 = 0$.

5. (a) $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. (b) $(1+i)^8 = (\sqrt{2})^8(\cos 2\pi + i \sin 2\pi) = 16(1+0i) = 16$.

6. (a) Product of conjugate pair: $(2+i)(2-i) = 5$. Product of all roots $= 320$ (constant term), so $\alpha^3 = 320/5 = 64 \Rightarrow \alpha = 4$. (b) Sum of roots $= (2+i) + (2-i) + 4 + 16 = 24 = -p \Rightarrow p = -24$.

7. (a) $z_1 z_2 = (2+bi)\left[(4-b^2) - 4bi\right] = (8+2b^2) - (b^3+4b)i$. (b) Noting $z_2 = \overline{z_1^2}$, we have $z_1 z_2 = z_1 \overline{z_1}^2 = |z_1|^2 \overline{z_1}$, so $\arg(z_1 z_2) = -\arg(z_1) = -\arctan\left(\frac{b}{2}\right)$. Setting this equal to $\frac{\pi}{4}$: $\arctan\left(\frac{b}{2}\right) = -\frac{\pi}{4} \Rightarrow \frac{b}{2} = -1 \Rightarrow b = -2$.

8. (a) Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then $z_1\overline{z_2} = (x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)$, so $\operatorname{Re}(z_1\overline{z_2}) = x_1x_2 + y_1y_2$, which is the dot product of the position vectors (x_1, y_1) and (x_2, y_2). This dot product is zero if and only if the vectors are perpendicular (given both are non-zero). (b) Require $3x_2 + 4y_2 = 0 \Rightarrow x_2 = -\frac{4y_2}{3}$. Letting $y_2 = 3k$, $x_2 = -4k$: $z_2 = k(-4+3i)$, $k \in \mathbb{R} \setminus \{0\}$.

9. (a) Sum of roots $= (a+bi) + (a-bi) + (b+ai) + (b-ai) = 2a + 2b$. This equals 8 (from $-(-8)$), so $a + b = 4$. (b) Product of roots (constant term) $= 100$. The roots pair as $(a+bi)(a-bi) = a^2 + b^2$ and $(b+ai)(b-ai) = a^2 + b^2$, so the product is $(a^2+b^2)^2 = 100 \Rightarrow a^2 + b^2 = 10$. (c) $(a+b)^2 = 16 = a^2 + 2ab + b^2 = 10 + 2ab \Rightarrow ab = 3$. So a, b are roots of $t^2 - 4t + 3 = 0 \Rightarrow (t-1)(t-3) = 0$, giving $a = 1$ or $a = 3$.

10. (a) Since $z_2 = z_1 e^{-i\pi/3}$, $z_2^2 = z_1^2 e^{-i2\pi/3}$, so $z_1^2 + z_2^2 = z_1^2\left(1 + e^{-i2\pi/3}\right)$. Now $e^{-i2\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, so $1 + e^{-i2\pi/3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-i\pi/3}$. Hence $z_1^2 + z_2^2 = z_1^2 e^{-i\pi/3} = z_1\left(z_1 e^{-i\pi/3}\right) = z_1 z_2$. (b) By Vieta's formulas, $z_1 + z_2 = -a$ and $z_1 z_2 = b$. Then $z_1^2 + z_2^2 = (z_1+z_2)^2 - 2z_1z_2 = a^2 - 2b$. By part (a), $z_1^2 + z_2^2 = z_1 z_2 = b$, so $a^2 - 2b = b \Rightarrow a^2 = 3b$. (c) With $a = 6$: $36 = 3b \Rightarrow b = 12$.