The Binomial Theorem

Worksheet 3 — Set C

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

Expand $(x+2)^4$ fully, simplifying each term. **[4]**

2. [Maximum mark: 5] *Difficulty:* 1/5

(a) Write down the value of **(i)** $\binom{10}{2}$, **(ii)** $\binom{6}{1}$, **(iii)** $\binom{4}{4}$. **[3]**

(b) Hence state the coefficient of x^2 in the expansion of $(1+x)^{10}$. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Consider the expansion of $(x+2)^8$.

(a) State the value of k for which the term $\binom{8}{k}x^{8-k}2^k$ contains x^5. **[2]**

(b) Hence find the term containing x^5, and state its coefficient. **[3]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Consider the expansion of $(3x-1)^6$.

(a) Find the term containing x^4. **[3]**

(b) Find the constant term. **[3]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Consider the binomial expansion

$$(x+1)^{10} = x^{10} + ax^9 + bx^8 + 120x^7 + \cdots + 1,$$

where $a, b \in \mathbb{Z}^+$.

(a) Find the value of a. **[3]**

(b) Find the value of b. **[3]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Find the coefficient of x^3 in the expansion of $(2x-5)^6$, giving your answer as an integer.

(a) State the value of k required, and write down the general term of the expansion. **[3]**

(b) Hence evaluate the coefficient of x^3. **[4]**

7. [Maximum mark: 7] *Difficulty:* 3/5

In the expansion of $(1+x)^n$, the coefficient of x^4 is four times the coefficient of x^3.

(a) Show that $\dfrac{n-3}{4}=4$. **[4]**

(b) Hence find the value of n. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Consider the expansion of $(cx+1)^4$, where $c \neq 0$. The coefficient of the x^2 term is 24.

(a) Show that $6c^2 = 24$. **[4]**

(b) Hence find the possible value(s) of c. **[4]**

9. [Maximum mark: 9] *Difficulty:* 4/5

In the expansion of $(1+x)^n$, the coefficients of the 9th, 10th and 11th terms are in arithmetic progression.

(a) By writing $2\binom{n}{9}=\binom{n}{8}+\binom{n}{10}$ and simplifying, show that $n^2-37n+322=0$. **[6]**

(b) Solve this equation, and state whether both roots are valid, given that $n \geq 10$. **[3]**

10. [Maximum mark: 11] *Difficulty:* 5/5

Challenge question. Consider the binomial expansion of $\sqrt{1+4x}$, i.e. $(1+4x)^{1/2}$.

(a) Find the first four terms, in ascending powers of x, of this expansion. **[5]**

(b) State the values of x for which the expansion is valid. **[2]**

(c) By substituting $x=0.01$, use your expansion to find an approximate value of $\sqrt{1.04}$, correct to five decimal places. Compare this with the value given by a calculator. **[4]**

Answer Key

1. $(x+2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$.

2. (a) (i) 45 (ii) 6 (iii) 1. (b) Coefficient of $x^2 = \binom{10}{2} = 45$.

3. (a) $8-k=5 \Rightarrow k=3$. (b) $\binom{8}{3}x^5(2)^3 = 56(8)x^5 = 448x^5$; coefficient $= 448$.

4. (a) $6-k=4 \Rightarrow k=2$: $\binom{6}{2}(3x)^4(-1)^2 = 15(81x^4)(1) = 1215x^4$. (b) $k=6$: $\binom{6}{6}(3x)^0(-1)^6 = 1$.

5. (a) $a = \binom{10}{1} = 10$. (b) $b = \binom{10}{2} = 45$ (consistent with $\binom{10}{3} = 120$ given).

6. (a) $6-k=3 \Rightarrow k=3$; general term $\binom{6}{3}(2x)^3(-5)^3$. (b) $\binom{6}{3}(2)^3(-5)^3 = 20(8)(-125) = -20000$.

7. (a) $\dfrac{\binom{n}{4}}{\binom{n}{3}} = \dfrac{n-3}{4} = 4$. (b) $n-3=16 \Rightarrow n=19$.

8. (a) The x^2 term arises from $k=2$: $\binom{4}{2}(cx)^2 = 6c^2x^2$, so $6c^2 = 24$. (b) $c^2 = 4 \Rightarrow c = \pm 2$.

9. (a) Dividing $2\binom{n}{9} = \binom{n}{8} + \binom{n}{10}$ through by $\binom{n}{8}$, using $\binom{n}{9}/\binom{n}{8} = \dfrac{n-8}{9}$ and $\binom{n}{10}/\binom{n}{8} = \dfrac{(n-8)(n-9)}{90}$: $\dfrac{2(n-8)}{9} = 1 + \dfrac{(n-8)(n-9)}{90}$. Multiplying by 90: $20(n-8) = 90 + (n-8)(n-9) \Rightarrow 20n - 160 = 90 + n^2 - 17n + 72 \Rightarrow n^2 - 37n + 322 = 0$. (b) $n = \dfrac{37 \pm 9}{2} = 23$ or 14. Both satisfy $n \geq 10$, so both values are valid.

10. (a) Using $(1+u)^k \approx 1 + ku + \dfrac{k(k-1)}{2!}u^2 + \dfrac{k(k-1)(k-2)}{3!}u^3$ with $k = \dfrac{1}{2}$, $u = 4x$: $(1+4x)^{1/2} = 1 + 2x - 2x^2 + 4x^3 + \ldots$ (b) Valid for $|4x| < 1 \Rightarrow |x| < \dfrac{1}{4}$. (c) With $x = 0.01$: $1 + 2(0.01) - 2(0.01)^2 + 4(0.01)^3 = 1.01980$. Calculator value: $\sqrt{1.04} \approx 1.01980$ — the approximation agrees to five decimal places.