The Binomial Theorem

Worksheet 2 — Set B

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

Expand $(3x+1)^4$ fully, simplifying each term. **[4]**

2. [Maximum mark: 5] *Difficulty:* 1/5

(a) Write down the value of **(i)** $\binom{7}{3}$, **(ii)** $\binom{9}{1}$, **(iii)** $\binom{6}{6}$. **[3]**

(b) Hence state the coefficient of x^3 in the expansion of $(1+x)^7$. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Consider the expansion of $(x+3)^7$.

(a) State the value of k for which the term $\binom{7}{k}x^{7-k}3^k$ contains x^4. **[2]**

(b) Hence find the term containing x^4, and state its coefficient. **[3]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Consider the expansion of $(2x-1)^6$.

(a) Find the term containing x^3. **[3]**

(b) Find the constant term. **[3]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Consider the binomial expansion

$$(x+1)^8 = x^8 + ax^7 + bx^6 + 56x^5 + \cdots + 1,$$

where $a, b \in \mathbb{Z}^+$.

(a) Find the value of a. **[3]**

(b) Find the value of b. **[3]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Find the coefficient of x^4 in the expansion of $(3x-2)^7$, giving your answer as an integer.

(a) State the value of k required, and write down the general term of the expansion. **[3]**

(b) Hence evaluate the coefficient of x^4. **[4]**

7. [Maximum mark: 7] *Difficulty:* 3/5

In the expansion of $(1+x)^n$, the coefficient of x^3 is three times the coefficient of x^2.

(a) Show that $\dfrac{n-2}{3} = 3$. **[4]**

(b) Hence find the value of n. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Consider the expansion of $(bx+1)^5$, where $b \neq 0$. The coefficient of the x^3 term is 270.

(a) Show that $10b^3 = 270$. **[4]**

(b) Hence find the value of b, explaining why it is unique. **[4]**

9. [Maximum mark: 9] *Difficulty:* 4/5

In the expansion of $(1+x)^n$, the coefficients of the 5th, 6th and 7th terms are in arithmetic progression.

(a) By writing $2\binom{n}{5} = \binom{n}{4} + \binom{n}{6}$ and simplifying, show that $n^2 - 21n + 98 = 0$. **[6]**

(b) Solve this equation, and state whether both roots are valid, given that $n \geq 6$. **[3]**

10. [Maximum mark: 11] *Difficulty:* 5/5

Challenge question. Consider the binomial expansion of $(1+3x)^{-2}$.

(a) Find the first four terms, in ascending powers of x, of this expansion. **[5]**

(b) State the values of x for which the expansion is valid. **[2]**

(c) By substituting $x = 0.01$, use your expansion to find an approximate value of $(1.03)^{-2}$, correct to five decimal places. Compare this with the value given by a calculator. **[4]**

Answer Key

1. $(3x+1)^4 = 81x^4 + 108x^3 + 54x^2 + 12x + 1$.

2. (a) (i) 35 (ii) 9 (iii) 1. (b) Coefficient of $x^3 = \binom{7}{3} = 35$.

3. (a) $7-k=4 \Rightarrow k=3$. (b) $\binom{7}{3}x^4(3)^3 = 35(27)x^4 = 945x^4$; coefficient $= 945$.

4. (a) $6-k=3 \Rightarrow k=3$: $\binom{6}{3}(2x)^3(-1)^3 = 20(8x^3)(-1) = -160x^3$. (b) $k=6$: $\binom{6}{6}(2x)^0(-1)^6 = 1$.

5. (a) $a = \binom{8}{1} = 8$. (b) $b = \binom{8}{2} = 28$ (consistent with $\binom{8}{3} = 56$ given).

6. (a) $7-k=4 \Rightarrow k=3$; general term $\binom{7}{3}(3x)^4(-2)^3$. (b) $\binom{7}{3}(3)^4(-2)^3 = 35(81)(-8) = -22680$.

7. (a) $\frac{\binom{n}{3}}{\binom{n}{2}} = \frac{n-2}{3} = 3$. (b) $n-2=9 \Rightarrow n=11$.

8. (a) The x^3 term arises from $k=2$: $\binom{5}{2}(bx)^3 = 10b^3x^3$, so $10b^3 = 270$. (b) $b^3 = 27 \Rightarrow b=3$; since cube roots of a positive real number are unique over $\mathbb{R}$, this is the only real solution.

9. (a) Dividing $2\binom{n}{5} = \binom{n}{4} + \binom{n}{6}$ through by $\binom{n}{4}$, and using $\binom{n}{5}/\binom{n}{4} = \frac{n-4}{5}$ and $\binom{n}{6}/\binom{n}{4} = \frac{(n-4)(n-5)}{30}$: $\frac{2(n-4)}{5} = 1 + \frac{(n-4)(n-5)}{30}$. Multiplying by 30: $12(n-4) = 30 + (n-4)(n-5) \Rightarrow 12n - 48 = 30 + n^2 - 9n + 20 \Rightarrow n^2 - 21n + 98 = 0$. (b) $n = \frac{21 \pm 7}{2} = 14$ or 7. Both satisfy $n \geq 6$, so both values are valid.

10. (a) Using $(1+u)^k \approx 1 + ku + \frac{k(k-1)}{2!}u^2 + \frac{k(k-1)(k-2)}{3!}u^3$ with $k=-2$, $u=3x$: $(1+3x)^{-2} = 1 - 6x + 27x^2 - 108x^3 + \dots$ (b) Valid for $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$. (c) With $x = 0.01$: $1 - 6(0.01) + 27(0.01)^2 - 108(0.01)^3 = 0.94259$. Calculator value: $(1.03)^{-2} \approx 0.94260$ — the approximation agrees closely, to within 0.00001.