The Binomial Theorem

Worksheet 1 — Set A

Mathematics: Analysis and Approaches HL

Instructions

- Answer all questions in the spaces provided. Show all working clearly: correct answers without working may not receive full marks.
- Unless told otherwise, give numerical answers exactly or correct to three significant figures.
- Questions are arranged in order of increasing difficulty. Question 10 on each worksheet is a *challenge question*, set beyond typical examination level.
- A calculator may be used unless a question indicates otherwise.

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1. [Maximum mark: 4] *Difficulty:* 1/5

Expand $(2x+1)^4$ fully, simplifying each term. **[4]**

2. [Maximum mark: 5] *Difficulty:* 1/5

(a) Write down the value of **(i)** $\binom{6}{2}$, **(ii)** $\binom{8}{0}$, **(iii)** $\binom{5}{5}$. **[3]**

(b) Hence state the coefficient of x^2 in the expansion of $(1+x)^6$. **[2]**

3. [Maximum mark: 5] *Difficulty:* 2/5

Consider the expansion of $(x+2)^6$.

(a) State the value of k for which the term $\binom{6}{k}x^{6-k}2^k$ contains x^3. **[2]**

(b) Hence find the term containing x^3, and state its coefficient. **[3]**

4. [Maximum mark: 6] *Difficulty:* 2/5

Consider the expansion of $(3x-2)^5$.

(a) Find the term containing x^2. **[3]**

(b) Find the constant term. **[3]**

5. [Maximum mark: 6] *Difficulty:* 2/5

Consider the binomial expansion

$$(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \cdots + 1,$$

where $a, b \in \mathbb{Z}^+$.

(a) Find the value of a. **[3]**

(b) Find the value of b. **[3]**

6. [Maximum mark: 7] *Difficulty:* 3/5

Find the coefficient of x^5 in the expansion of $(2x-3)^8$, giving your answer as an integer.

(a) State the value of k required, and write down the general term of the expansion. **[3]**

(b) Hence evaluate the coefficient of x^5. **[4]**

7. [Maximum mark: 7] *Difficulty:* 3/5

In the expansion of $(1+x)^n$, the coefficient of x^4 is three times the coefficient of x^3.

(a) Show that $\dfrac{n-3}{4}=3$. **[4]**

(b) Hence find the value of n. **[3]**

8. [Maximum mark: 8] *Difficulty:* 4/5

Consider the expansion of $(ax+1)^6$, where $a \neq 0$. The coefficient of the x^2 term is 60.

(a) Show that $15a^2 = 60$. **[4]**

(b) Hence find the possible value(s) of a. **[4]**

9. [Maximum mark: 9] *Difficulty:* 4/5

In the expansion of $(1+x)^n$, the coefficients of the 2nd, 3rd and 4th terms are in arithmetic progression.

(a) By writing $2\binom{n}{2} = \binom{n}{1} + \binom{n}{3}$ and simplifying, show that $n^2 - 9n + 14 = 0$. **[6]**

(b) Solve this equation, and explain why one of the two roots must be rejected. **[3]**

10. [Maximum mark: 11] *Difficulty:* 5/5

Challenge question. Consider the binomial expansion of $(1-2x)^{-3}$.

(a) Find the first four terms, in ascending powers of x, of this expansion. **[5]**

(b) State the values of x for which the expansion is valid. **[2]**

(c) By substituting $x = 0.01$, use your expansion to find an approximate value of $(0.98)^{-3}$, correct to five decimal places. Compare this with the value given by a calculator. **[4]**

Answer Key

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1. $(2x+1)^4 = 16x^4+32x^3+24x^2+8x+1$.

2. (a) (i) 15 (ii) 1 (iii) 1. (b) Coefficient of $x^2 = \binom{6}{2} = 15$.

3. (a) $6-k=3 \Rightarrow k=3$. (b) $\binom{6}{3}x^3(2)^3 = 20(8)x^3 = 160x^3$; coefficient $= 160$.

4. (a) $6-k=2 \Rightarrow k=3$ (using base $3x$, exponent 2 on -2): $\binom{5}{3}(3x)^2(-2)^3 = 10(9x^2)(-8) = -720x^2$. (b) $k=5$: $\binom{5}{5}(3x)^0(-2)^5 = -32$.

5. (a) $a = \binom{7}{1} = 7$. (b) $b = \binom{7}{2} = 21$ (consistent with $\binom{7}{3} = 35$ given).

6. (a) $8-k=5 \Rightarrow k=3$; general term $\binom{8}{3}(2x)^5(-3)^3$. (b) $\binom{8}{3}(2)^5(-3)^3 = 56(32)(-27) = -48384$.

7. (a) $\dfrac{\binom{n}{4}}{\binom{n}{3}} = \dfrac{n-3}{4} = 3$. (b) $n-3=12 \Rightarrow n=15$.

8. (a) The x^2 term arises from $k=4$: $\binom{6}{4}(ax)^2 = 15a^2x^2$, so $15a^2 = 60$. (b) $a^2=4 \Rightarrow a=\pm 2$.

9. (a) $2\binom{n}{2} = \binom{n}{1}+\binom{n}{3} \Rightarrow n(n-1) = n+\dfrac{n(n-1)(n-2)}{6}$. Dividing by n and simplifying: $6(n-1) = 6+(n-1)(n-2) \Rightarrow n^2-9n+14=0$. (b) $n = \dfrac{9\pm 5}{2} = 7$ or 2. Since $n=2$ does not admit a 4th term (there is no $\binom{2}{3}$ term), it is rejected; so $n=7$.

10. (a) Using $(1+u)^k \approx 1+ku+\dfrac{k(k-1)}{2!}u^2+\dfrac{k(k-1)(k-2)}{3!}u^3$ with $k=-3$, $u=-2x$: $(1-2x)^{-3} = 1+6x+24x^2+80x^3+\dots$ (b) Valid for $|-2x|<1 \Rightarrow |x|<\dfrac{1}{2}$. (c) With $x=0.01$: $1+6(0.01)+24(0.01)^2+80(0.01)^3 = 1.06248$. Calculator value: $(0.98)^{-3} \approx 1.06248$ — the approximation agrees to five decimal places.